# Application of monotone type operators to nonlinear PDE's

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# Contents

# Chapter 1

# NONLINEAR STATIONARY PROBLEMS

## 1 Introduction

The aim of these lecture notes is to give a short itroduction to the theory of monotone type operators, and by using this theory to consider abstract stationary and evolution equations with operators of this type. Then the abstract theory will be applied to “weak” solutions of nonlinear elliptic, parabolic, functional parabolic, hyperbolic and functional hyperbolic equations of “divergence type”. By using the theory of monotone type operators, it is possible to treat several types of nonlinear partial differential equations (not only semilinear PDEs) and to prove global existence of solutions of time dependent problems. However, there are a lot of problems in physics, chemistry, biology etc. the mathematical models of which are nonlinear PDEs but the monotone type operators can not be applied to them. These equations need particular treatment. (see, e.g. [13], [18], [22], [23], [36], [38], [52], [65], [67]).

The lecture notes are based mainly on the theory of second order linear partial differential equations (see, e.g., [67], [27]), some fundamental notions and theorems of functional analysis (see, e.g., [42], [66], [92], [8]) and the theory of ordinary differential equations (see, e.g., [88], [19], [35]). The importance of linear and nonlinear partial differential equations in physical, chemical, biological etc. applications is well known (see, e.g., the above references). The classical results on linear and quasilinear second order partial differential equations can be found in the monographs [28], [31], [37], [44], [51], [43], [49] and also in the books [7], [27], [60], [62], [64], [67], [89].

Partial functional differential equations arise in biology, chemistry, physics, climatology (see, e.g., [13], [18], [21]–[23], [36], [38], [52], [65], [91] and the references therein). The systematic study of such equations from the dynamical system and semigroup point of view began in the 70s. Several results in this direction can be found in the monographs [60], [89], [91]. This approach is mostly based on arguments used in the theory of ordinary differential equations and functional differential equations (see [24], [32]–[34], [58], [59]).

In the classical work [50] of J.L. Lions one can find the fundamental results on monotone type operators and their applications to nonlinear partial differential equations. Further important monographs have been written by E. Zeidler [93] and H. Gajewski, K. Gröger, K. Zacharias [30], S.Fučik, A. Kufner in [29]. A good summary of further results on monotone type operators, based on degree theory (see, e.g., [20]) and its applications to nonlinear evolution equations is in the works [8] and [57] of V. Mustonen and J. Berkovits. By using the theory of monotone type operators one obtains directly the global existence of weak solutions, also for higher order nonlinear partial differential equations, satisfying certain conditions which are more restrictive (in some sense) than in the case of the previous approach.

It turned out that one can apply the theory of monotone type operators (e.g. pseudomonotone operators) to nonlinear elliptic variational inequalities, further, to nonlinear parabolic and certain hyperbolic functional differential equations and systems to get existence and uniqueness theorems on weak solutions and results on qualitative properties of weak solutions, including, e.g., "strongly nonlinear" and "non-uniformly" parabolic equations.

In Chapter 1 we shall consider nonlinear stationary problems and as particular cases nonlinear elliptic differential equations, functional equations and variational inequalities. In Chapter 2 first order evolution equations and as particular cases nonlinear parabolic differential equations, functional parabolic equations will be considered. Finally, in Chapter 3 second order nonlinear evolution equations and certain nonlinear hyperbolic equations will be treated. In each chapter the "general" results are illustrated by examples.

In this section we shall give a motivation of the abstract stationary problem and we shall formulate it, by using the definition of the "weak" ("generalized") solution to boundary value problems for nonlinear elliptic equations of "divergence type".

First we recall the definition of the weak solution to the linear elliptic equation of the form

$$-\sum_{j,k=1}^{n} D_j(a_{jk}D_k u) + cu = f \text{ in the bounded domain } \Omega \subset \mathbb{R}^n \tag{1.1}$$

($D_j = \frac{\partial}{\partial x_j}$) with the Dirichlet boundary condition

$$u|_{\partial\Omega} = \varphi. \tag{1.2}$$

Assuming that $u$ is a sufficiently smooth (for simplicity, e.g. $u \in C^2(\overline{\Omega})$) solution of (1.1), (1.2) and $\partial\Omega$ is sufficiently smooth (e.g. $C^1$ or in some sense piecewise $C^1$ surface), multiply (1.1) by a test function $v \in C_0^1(\Omega)$ and integrate over $\Omega$, by using Gauss's theorem, we obtain

$$\sum_{j,k=1}^{n} \int_\Omega a_{jk}(D_k u)(D_j v) + \int_\Omega cuv = \int_\Omega fv. \tag{1.3}$$

Assuming $a_{jk}, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$, (1.3) holds for arbitrary element $v$ of the Sobolev space $H_0^1(\Omega)$ (See, e.g. [67].) Therefore, weak solution of the Dirichlet problem (1.1), (1.2) is defined as a function $u \in H^1(\Omega)$, satisfying (1.3) for all $v \in H_0^1(\Omega)$ and the boundary condition (1.2) where $u|_{\partial\Omega}$ means the trace of $u \in H^1(\Omega)$. In the particular case when $\varphi = 0$, the weak solution of (1.3) is a function $u \in H_0^1(\Omega)$.

Thus every classical solution $u \in C^2(\overline{\Omega})$ of (1.1), (1.2) is a weak solution and it is not difficult to show that if $u$ is a weak solution and it is sufficiently smooth (e.g. $u \in C^2(\overline{\Omega})$), then $u$ is a classical solution, too. The details of the above arguments can be found e.g. in [67], [44].

The weak solution of the nonlinear equation of "divergence form"

$$-\sum_{j=1}^n D_j[a_j(x, u(x), Du(x))] + a_0(x, u(x), Du(x)) = f(x) \text{ for all } x \in \Omega \tag{1.4}$$

($Du = (D_1 u, ..., D_n u)$) with the Dirichlet boundary condition (1.2) is defined in a similar way. Assume that $u$ is a classical (sufficiently smooth) solution of (1.4). Multiply the equation (1.4) by a test function $v \in C_0^1(\Omega)$ and integrate over $\Omega$. By Gauss's theorem we obtain

$$\sum_{j=1}^n \int_\Omega a_j(x, u(x), Du(x)) D_j v(x) dx + \int_\Omega a_0(x, u(x), Du(x)) v(x) dx = \tag{1.5}$$

$$\int_\Omega f(x) v(x) dx.$$

Later we shall see that if the functions $a_j$ satisfy a certain growth condition (see later Condition $A_2$) then for an arbitrary element $u$ of the Sobolev space $W^{1,p}(\Omega)$ ($1 < p < \infty$) (see the definition e.g. in [67], [1], [93]) we have $a_j(x, u.Du) \in L^q(\Omega)$ where $1/p + 1/q = 1$. Consequently, (1.5) holds for all test functions $v \in W_0^{1,p}(\Omega)$ because $W_0^{1,p}(\Omega)$ is the closure of $C_0^1(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$.

Thus, similarly to the linear case, the weak solution of (1.4), (1.2) is defined as a function $u \in W^{1,p}(\Omega)$ satisfying (1.5) for all $v \in W_0^{1,p}(\Omega)$ and (1.2) where $u|_{\partial\Omega}$ denotes the trace of $u \in W^{1,p}(\Omega)$ on $\partial\Omega$. In the particular case $\varphi = 0$ the weak solution is a function $u \in W_0^{1,p}(\Omega)$ satisfying (1.5) for all $v \in W_0^{1,p}(\Omega)$. Similarly to the linear case, a sufficiently smooth function $u$ is a classical solution if and only if it is a weak solution.

Assume that the functions $a_j$ fulfil the above mentioned growth condition such that $a_j(x, u, Du) \in L^q(\Omega)$ for all $u \in W_0^{1,p}(\Omega)$. Then equation (1.5), i.e. the fact that $u$ is a weak solution (in the case $\varphi = 0$) can be interpreted in the following way. Denote the left hand side of (1.5) by $\langle A(u), v\rangle$, i.e.

$$\langle A(u), v\rangle = \sum_{j=1}^n \int_\Omega a_j(x, u(x), Du(x)) D_j v(x) dx + \tag{1.6}$$

$$\int_\Omega a_0(x, u(x), Du(x))v(x)dx.$$

For a fixed $u \in W_0^{1,p}(\Omega)$, $\langle A(u), v\rangle$ is a linear continuous functional applied to $v \in W_0^{1,p}(\Omega)$, i.e. $A(u)$ belongs to the dual space of $V = W_0^{1,p}(\Omega)$. Thus, according to (1.6), we have a (nonlinear) operator $A : V \to V^\star$. Further, by using the notation

$$\langle F, v\rangle = \int_\Omega f(x)v(x)dx, \tag{1.7}$$

we have $F \in V^\star$ if $f \in L^q(\Omega)$.

Summarizing, in the case $\varphi = 0$ one may write (1.5) in the abstract form

$$A(u) = F \tag{1.8}$$

where $A : V \to V^\star$ is a nonlinear operator and $F$ is a given element of $V^\star$.

In Section 3 we shall show that in the case $V = W^{1,p}(\Omega)$ equation (1.8) is an abstract form of weak formulation of (1.4) with a Neumann type homogeneous boundary condition.

In the next section we shall formulate and prove existence and uniqueness theorems regarding (1.8), by using the theory of monotone type operators.

## 2 Existence and uniqueness theorems

First we formulate some basic definitions for (possibly nonlinear) operators $A : V \to V^\star$. Denote by $V$ a real Banach space and $V^\star$ its dual space.

**Definition 2.1.** *Operator $A : V \to V^\star$ is called* bounded *if it maps bounded sets of $V$ into bounded sets of $V^\star$.*

**Definition 2.2.** *Operator $A : V \to V^\star$ is said to be* hemicontinuous *if for each fixed $u_1, u_2, v \in V$ the function*

$$\lambda \mapsto \langle A(u_1 + \lambda u_2), v\rangle, \quad \lambda \in \mathbb{R} \text{ is continuous.}$$

**Definition 2.3.** *Operator $A : V \to V^\star$ is said to be* monotone *if*

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq 0 \text{ for all } u_1, u_2 \in V.$$

*If for $u_1 \neq u_2$*

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle > 0,$$

*$A$ is said to be* strictly monotone.

**Definition 2.4.** *A bounded operator $A : V \to V^\star$ is said to be* pseudomonotone *if*

$$(u_j) \to u \text{ weakly in } V, \quad \limsup_{j\to\infty}\langle A(u_j), u_j - u\rangle \leq 0 \tag{2.1}$$

*imply*

$$\lim_{j\to\infty} \langle A(u_j), u_j - u\rangle = 0 \text{ and } (A(u_j)) \to A(u) \text{ weakly in } V^\star. \tag{2.2}$$

**Proposition 2.5.** *Let $V$ be a reflexive Banach space. Assume that $A : V \to V^\star$ is bounded, hemicontinuous and monotone. Then $A$ is pseudomonotone.*

*Proof.* Assume (2.1). Since $A$ is monotone,

$$\langle A(u_j) - A(u), u_j - u\rangle \geq 0,$$

hence

$$\langle A(u_j), u_j - u\rangle \geq \langle A(u), u_j - u\rangle. \tag{2.3}$$

By (2.1), we have

$$\lim_{j\to\infty} \langle A(u), u_j - u\rangle = 0, \tag{2.4}$$

thus (2.1), (2.3), (2.4) imply

$$\lim_{j\to\infty} \langle A(u_j), u_j - u\rangle = 0. \tag{2.5}$$

In order to show the second part of (2.2) consider

$$w = (1-\lambda)u + \lambda v \tag{2.6}$$

with arbitrary $v \in V$ and $\lambda > 0$. Since $A$ is monotone,

$$\langle A(u_j) - A(w), u_j - w\rangle \geq 0,$$

whence

$$\langle A(u_j), u_j - u\rangle + \langle A(u_j), u - w\rangle - \langle A(w), u_j - u\rangle - \langle A(w), u - w\rangle \geq 0$$

or equivalently

$$\langle A(u_j), u_j - u\rangle + \langle A(u_j), \lambda(u - v)\rangle - \langle A(w), u_j - u\rangle - \langle A(w), \lambda(u - v)\rangle \geq 0. \tag{2.7}$$

By (2.1),

$$\lim_{j\to\infty} \langle A(w), u_j - u\rangle = 0$$

and so (2.5), (2.7) imply

$$\liminf_{j\to\infty} \langle A(u_j), \lambda(u - v)\rangle \geq \langle A(w), \lambda(u - v)\rangle,$$

thus, due to $\lambda > 0$

$$\liminf_{j\to\infty} \langle A(u_j), u - v\rangle \geq \langle A(w), u - v\rangle = \langle A((1-\lambda)u + \lambda v), u - v\rangle. \tag{2.8}$$

Since $A$ is hemicontinuous, as $\lambda \to +0$ we obtain from (2.8)

$$\liminf_{j\to\infty} \langle A(u_j), (u-v)\rangle \geq \langle A(u), (u-v)\rangle. \tag{2.9}$$

The sequence $(A(u_j))$ is bounded in $V^\star$, so there is a subsequence $(A(u_{j_k}))$ of $(A(u_j))$ which is weakly convergent to some $\chi \in V^\star$, thus from (2.9) we obtain

$$\langle \chi, u-v\rangle \geq \langle A(u), u-v\rangle. \tag{2.10}$$

As (2.10) holds for arbitrary $v \in V$, it follows $\chi = A(u)$. Thus the second part of (2.2) holds for a subsequence of $(u_j)$. We show that it must hold for the whole sequence, by using the following trick.

**Cantor's trick** Assume the contrary. Then there exist $\varepsilon_0 > 0$, a subsequence $(\tilde{u}_j)$ of $(u_j)$ and $v \in V$ such that

$$|\langle A(\tilde{u}_j) - A(u), v\rangle| \geq \varepsilon_0. \tag{2.11}$$

Applying the above argument to the sequence $(\tilde{u}_j)$ (instead of $(u_j)$), we obtain a subsequence $(\tilde{u}_{j_k})$ of $(\tilde{u}_j)$ for which

$$(A(\tilde{u}_{j_k})) \to A(u) \text{ weakly in } V^\star$$

which contradicts to (2.11). □

**Definition 2.6.** *Operator $A : V \to V^\star$ is called* demicontinuous *if*

$$(u_j) \to u \textit{ strongly in } V \textit{ implies } (A(u_j)) \to A(u) \textit{ weakly in } V^\star.$$

**Proposition 2.7.** *If a bounded operator $A : V \to V^\star$ is pseudomonotone then $A$ is demicontinuous.*

*Proof.* Assume that $(u_j) \to u$ strongly in $V$. Then

$$|\langle A(u_j), u_j - u\rangle| \leq \|A(u_j)\|_{V^\star} \|u_j - u\|_V \to 0$$

because $\|A(u_j)\|_{V^\star}$ is bounded. Since $A$ is pseudomonotone,

$$(A(u_j)) \to A(u) \text{ weakly in } V^\star.$$

□

**Definition 2.8.** *Operator $A : V \to V^\star$ is called* belonging to $(S)_+$ *if*

$$(u_j) \to u \textit{ weakly in } V, \quad \limsup \langle A(u_j), u_j - u\rangle \leq 0$$

*imply $(u_j) \to u$ strongly in $V$.*

From definitions 2.4, 2.6, 2.8 immediately follows

**Proposition 2.9.** *If $A : V \to V^\star$ is bounded, demicontinuous and belongs to $(S)_+$ then $A$ is pseudomonotone.*

**Definition 2.10.** *Operator $A : V \to V^\star$ is called* coercive *if*

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle}{\|u\|} = +\infty.$$

**Remark 2.11.** If the linear operator $A$ is strictly positive in the sense that it satisfies

$$\langle Au, u\rangle \geq c\|u\|^2$$

with some constant $c > 0$ then $A$ is coercive.

Now consider the equation

$$A(u) = F \tag{2.12}$$

with an arbitrary $F \in V^\star$ where $A : V \to V^\star$ is a given (possibly nonlinear) operator. First we prove an existence theorem when $A$ is pseudomonotone. As a consequence, we shall obtain an existence and uniqueness theorem when $A$ is strictly monotone.

**Theorem 2.12.** *Let $V$ be a reflexive separable Banach space. Assume that $A : V \to V^\star$ is bounded, pseudomonotone and coercive. Then for arbitrary $F \in V^\star$ there exists a solution $u \in V$ of equation (2.12).*

The proof of this theorem is based on Galerkin's method and on the following lemma.

**Lemma 2.13.** *("acute angle lemma") Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and suppose: there exists $\rho > 0$ such that*

$$\langle g(\xi), \xi\rangle_{\mathbb{R}^n} \geq 0 \text{ for } |\xi| = \rho. \tag{2.13}$$

*Then there exists $\xi_0 \in \mathbb{R}^n$ such that*

$$g(\xi_0) = 0, \quad |\xi_0| \leq \rho. \tag{2.14}$$

*Proof.* We prove by contradiction. Assume that (2.14) is not true. Then $g(\xi) \neq 0$ for $|\xi| \leq \rho$ and thus

$$h(\xi) = -\rho\frac{g(\xi)}{|g(\xi)|}, \quad |\xi| \leq \rho$$

is a continuous function mapping the closed ball $\overline{B_\rho} = \{\xi \in \mathbb{R}^n : |\xi| \leq \rho\}$ into itself, because $|h(\xi)| = \rho$. By Brouwer's fixed point theorem $h$ has a fixed point $\xi^\star$, i.e.

$$h(\xi^\star) = \xi^\star, \quad |\xi^\star| = \rho.$$

Then

$$\langle h(\xi^\star), \xi^\star \rangle_{\mathbb{R}^n} = |\xi^\star|^2 = \rho^2 > 0$$

which is impossible since by (2.13)

$$\langle h(\xi^\star), \xi^\star \rangle_{\mathbb{R}^n} = \left\langle -\rho \frac{g(\xi^\star)}{|g(\xi^\star)|}, \xi^\star \right\rangle = -\frac{\rho}{|g(\xi^\star)|} \langle g(\xi^\star), \xi^\star \rangle_{\mathbb{R}^n} \leq 0.$$

□

*Proof of Theorem 2.12.* Since $V$ is separable, there exists a system $z_1, z_2, ...$ of linearly independent elements of $V$ such that their linear combinations are dense in $V$. Denote by $V_m$ the set of linear combinations of $z_1, z_2, \ldots, z_m$.

First by using Galerkin's approximation method, we construct the "$m$-th approximation" $u_m \in V_m$ of the solution $u \in V$ of (2.12) such that

$$\langle A(u_m), z_j \rangle = \langle F, z_j \rangle, \quad j = 1, \ldots, m, \tag{2.15}$$

or equivalently

$$\langle A(u_m), v \rangle = \langle F, v \rangle, \qquad \text{for } v \in V_m. \tag{2.16}$$

In order to do this, we apply Lemma 2.13 to the function $g = (g_1, g_2, \ldots, g_m)$, defined by

$$g_j(\xi_1, \ldots, \xi_m) = \langle A(\xi_1 z_1 + \cdots + \xi_m z_m), z_j \rangle - \langle F, z_j \rangle, \quad \xi \in \mathbb{R}^m, \quad j = 1, \ldots, m.$$

Since $A$ is bounded and pseudomonotone, $A$ is demicontinuous by Proposition 2.7 which implies that the functions $g_j$ are continuous. Further, introducing $\sum_{j=1}^m \xi_j z_j$ by $y$ and assuming $y \neq 0$, we have

$$\langle g(\xi), \xi \rangle_{\mathbb{R}^m} = \sum_{j=1}^m g_j(\xi)\xi_j = \left\langle A(\sum_{j=1}^m \xi_j z_j), \sum_{j=1}^m \xi_j z_j \right\rangle - \left\langle F, \sum_{j=1}^m \xi_j z_j \right\rangle =$$

$$\left[ \frac{\langle A(y), y \rangle}{\|y\|_V} - \frac{\langle F, y \rangle}{\|y\|_V} \right] \|y\|_V \geq \left[ \frac{\langle A(y), y \rangle}{\|y\|_V} - \|F\|_{V^\star} \right] \|y\|_V.$$

Operator $A$ is coercive, hence

$$\lim_{\|y\| \to \infty} \frac{\langle A(y), y \rangle}{\|y\|_V} = +\infty,$$

thus the right-hand side is positive if $\|y\|_V$ is sufficiently large, which is satisfied if $|\xi|$ is sufficiently large. So by Lemma 2.13 there exists $\xi \in \mathbb{R}^m$ such that $g(\xi) = 0$, i.e. we have a solution $u_m$ of (2.15).

If $V$ is of finite dimension, Theorem 2.12 is proved. Consider the remaining case when $V$ is of infinite dimension. Then we have a sequence $(u_m)$ of elements satisfying (2.16). The coercivity of $A$ implies that $(u_m)$ is a bounded sequence in $V$. Indeed, assuming that $(u_m)$ is not bounded, we would have a subsequence $(u_{m_k})$ such that

$$\lim_{k \to \infty} \|u_{m_k}\|_V = \infty,$$

which is impossible because by (2.16)

$$0 = \langle A(u_{m_k}), u_{m_k}\rangle - \langle F, u_{m_k}\rangle \geq \left[\frac{\langle A(u_{m_k}), u_{m_k}\rangle}{\|u_{m_k}\|_V} - \|F\|_{V^\star}\right] \|u_{m_k}\|_V \to +\infty$$

as $k \to \infty$ since $A$ is coercive.

The operator $A : V \to V^\star$ is bounded thus the sequence $(A(u_m))$ is bounded in $V^\star$. Since $V$ is reflexive, there are $u \in V$, $\chi \in V^\star$ and a subsequence $(u_{m_k})$ of $(u_m)$ such that

$$(u_{m_k}) \to u \text{ weakly in } V \tag{2.17}$$

and

$$(A(u_{m_k})) \to \chi \text{ weakly in } V^\star. \tag{2.18}$$

Now we show that $\chi = F$. Due to (2.16), for arbitrary fixed finite linear combination $v$ of $z_1, z_2, \dots$

$$\langle A(u_{m_k}), v\rangle = \langle F, v\rangle \tag{2.19}$$

for sufficiently large $k$. From (2.18), (2.19) as $k \to \infty$ we obtain $\langle \chi, v\rangle = \langle F, v\rangle$ for any finite linear combination of $z_1, z_2, \ldots$. Since the finite linear combinations are dense in $V$, we find $\chi = F$.

Finally, pseudomonotonicity of $A$ implies $A(u) = \chi (= F)$. Indeed, according to (2.17), $(u_{m_k}) \to u$ weakly in $V$ and by (2.16), (2.18)

$$\langle A(u_{m_k}), u_{m_k} - u\rangle = \langle A(u_{m_k}), u_{m_k}\rangle - \langle A(u_{m_k}), u\rangle = \tag{2.20}$$

$$\langle F, u_{m_k}\rangle - \langle A(u_{m_k}), u\rangle \to \langle F, u\rangle - \langle \chi, u\rangle = 0 \text{ as } k \to \infty.$$

□

**Theorem 2.14.** *Let $V$ be a reflexive separable Banach space and assume that $A : V \to V^\star$ is bounded, hemicontinuous, monotone and coercive. Then for arbitrary $F \in V^\star$ there exists a solution $u \in V$ of (2.12). If $A$ is strictly monotone then the solution is unique.*

*Proof.* By Proposition 2.5 $A$ is pseudomonotone, thus Theorem 2.12 implies the existence of a solution $u \in V$ of (2.12). Assume that $A$ is strictly monotone and

$$A(u_j) = F \text{ for } j = 1, 2.$$

Then

$$0 = \langle A(u_1) - A(u_2), u_1 - u_2\rangle$$

whence $u_1 = u_2$. □

**Definition 2.15.** *Operator $A : V \to V^\star$ is said to be* uniformly monotone *if there exists a strictly monotone increasing continuous function*

$$a : [0, \infty) \to [0, \infty) \textit{ with } a(0) = 0, \quad \lim_{+\infty} a = +\infty$$

*such that*

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq a(\|u_1 - u_2\|_V)\|u_1 - u_2\|_V \text{ for all } u_1, u_2 \in V.$$

**Remark 2.16.** If $A$ is uniformly monotone then it is strictly monotone. Function $a$ may be chosen as $a(t) = ct^{p-1}$ with constants $c > 0, p > 1$.

**Remark 2.17.** If $A$ is uniformly monotone then

$$\|A(u_1) - A(u_2)\|_{V^\star} \geq a(\|u_1 - u_2\|_V), \quad u_1, u_2 \in V \tag{2.21}$$

because

$$\|A(u_1) - A(u_2)\|_{V^\star}\|u_1 - u_2\|_V \geq \langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq$$

$$a(\|u_1 - u_2\|_V)\|u_1 - u_2\|_V.$$

If (2.21) holds then operator $A$ is called stable. In this case the solution of the equation (2.12) is unique and the solution $u$ of (2.12) depends continuously on the right hand side $F \in V^\star$, because by (2.21)

$$\|u_1 - u_2\|_V \leq a^{-1}\left(\|A(u_1) - A(u_2)\|_{V^\star}\right),$$

$a^{-1} : [0, \infty) \to [0, \infty)$ is a continuous function and $a^{-1}(0) = 0$.

**Remark 2.18.** According to the proof of Theorem 2.12 the sequence $(u_n)$, constructed by Galerkin's method, contains a subsequence which converges weakly in $V$ to a solution $u$ of (2.12). If the solution of (2.12) is unique (e.g. if $A$ is strictly monotone) then also the sequence $(u_n)$ must converge to $u$. Indeed, assuming the contrary, one gets contradiction, by using Cantor's trick (see in the proof of Proposition 2.5).

If $A$ is uniformly monotone then $(u_n) \to u$ also with respect to the norm of $V$. Indeed, let $\tilde{a}(t) = a(t)t$ which clearly has the same properties as $a$, further,

$$\tilde{a}(\|u_n - u\|_V) = a(\|u_n - u\|_V)\|u_n - u\|_V \leq \langle A(u_n) - A(u), u_n - u \rangle =$$

$$\langle A(u_n), u_n - u \rangle - \langle A(u), u_n - u \rangle \to 0 \text{ as } u \to \infty$$

by (2.17), (2.20), hence

$$\lim_{n \to \infty} \|u_n - u\|_V = 0.$$

# 3 Application of monotone operators

Now we shall apply Theorem 2.14 to the case when $V$ is a closed linear subspace of the Sobolev space $W^{1,p}(\Omega)$, containing $W_0^{1,p}(\Omega)$ ($1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary). Further, the operator $A : V \to V^\star$ will be given by

$$\langle A(u), v\rangle = \sum_{j=1}^{n} \int_\Omega a_j(x, u(x), Du(x)) D_j v(x) dx + \tag{3.1}$$

$$\int_\Omega a_0(x, u(x), Du(x)) v(x) dx, \quad u, v \in V$$

where the functions $a_j : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy conditions which will imply the assumptions of Theorem 2.14.

$(A1)$ Assume that the functions $a_j : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy the Carathéodory conditions, i.e. for a.a. fixed $x \in \Omega$, the function $\xi \mapsto a_j(x, \xi)$, $\xi \in \mathbb{R}^{n+1}$ is continuous and for each fixed $\xi \in \mathbb{R}^{n+1}$, $x \mapsto a_j(x, \xi)$, $x \in \Omega$ is measurable.

$(A2)$ Assume that there exist a constant $c_1$ and a nonnegative function $k_1 \in L^q(\Omega)$ $(1/p + 1/q = 1)$ such that for a.a, $x \in \Omega$, each $\xi \in \mathbb{R}^{n+1}$

$$|a_j(x, \xi)| \le c_1 |\xi|^{p-1} + k_1(x).$$

**Proposition 3.1.** *Assume that conditions $(A1)$, $(A2)$ are satisfied. Then $A : V \to V^\star$ is bounded and hemicontinuous.*

*Proof.* By $(A1)$ the function $x \mapsto a_j(x, u(x), Du(x))$ is measurable for arbitrary $u \in V$. Further, by $(A2)$

$$\int_\Omega |a_j(x, u(x), Du(x))|^q dx \le$$

$$\text{const}\left[\int_\Omega |(u(x), Du(x))|^{(p-1)q} dx + \int_\Omega k_1(x)^q dx\right] \le \text{const}\,[\|u\|_V^p + 1]$$

and so Hölder's inequality implies

$$|\langle A(u), v\rangle| \le \sum_{j=1}^{n} \left[\int_\Omega |a_j(x, u, Du)|^q dx\right]^{1/q} \|D_j v\|_{L^p(\Omega)} + \tag{3.2}$$

$$\left[\int_\Omega |a_0(x, u, Du)|^q dx\right]^{1/q} \|v\|_{L^p(\Omega)} \le \text{const}\left[\|u\|_V^{p/q} + 1\right] \|v\|_V.$$

By (3.2) it follows that $A(u)$ is a bounded linear operator on $V$ and

$$\|A(u)\|_{V^\star} \le \text{const}\left[\|u\|_V^{p/q} + 1\right],$$

thus $A : V \to V^\star$ is bounded.

Now we show that $A$ is hemicontinuous. Consider with fixed $u_1, u_2, v \in V$ the function

$$\lambda \mapsto \langle A(u_1 + \lambda u_2), v\rangle, \quad \lambda \in \mathbb{R}.$$

For the operator $A$, given by (3.1) we have

$$\langle A(u_1 + \lambda u_2), v\rangle = \sum_{j=1}^{n} \int_\Omega a_j(x, u_1 + \lambda u_2, Du_1 + \lambda Du_2) D_j v dx +$$

$$\int_\Omega a_0(x, u_1 + \lambda u_2, Du_1 + \lambda Du_2) v dx.$$

Assume that $\lim_{k\to\infty} \lambda_k = \lambda$ for a sequence $(\lambda_k)$. Then by $(A1)$ for a.a. $x \in \Omega$

$$\lim_{k\to\infty} a_j(x, u_1+\lambda_k u_2, Du_1+\lambda_k Du_2) = a_j(x, u_1+\lambda u_2, Du_1+\lambda Du_2), \quad j = 0, 1, \ldots, n,$$

further, by $(A2)$

$$|a_j(x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)|^q \leq \tag{3.3}$$

$$\text{const}[|(u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)|^p + k_1(x)^q] \leq$$

$$\text{const}[|u_1|^p + |u_2|^p + |Du_1|^p + |Du_2|^p + k_1(x)^q]$$

because $(\lambda_k)$ is bounded. Thus by Young's inequality

$$|a_j(x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2) D_j v| \leq$$

$$\text{const}[|u_1|^p + |u_2|^p + |Du_1|^p + |Du_2|^p + k_1(x)^q] + \text{const}|D_j v|^p$$

and similar inequality holds for

$$|a_0(x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2) v|.$$

Thus by (3.3) Lebesgue's dominated convergence theorem implies

$$\lim_{k\to\infty} \langle A(u_1 + \lambda_k u_2), v\rangle = \langle A(u_1 + \lambda u_2), v\rangle$$

which completes the proof of Proposition 3.1. $\square$

Now we formulate assumptions which, clearly, imply that operator $A$, defined by (3.1) is monotone and coercive.

$(A3)$ Assume that for a.a. $x \in \Omega$, all $\xi, \xi^\star \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n} [a_j(x, \xi) - a_j(x, \xi^\star)](\xi_j - \xi_j^\star) \geq 0.$$

$(A4)$ Assume that there exist a constant $c_2 > 0$ and $k_2 \in L^1(\Omega)$ such that for a.a. $x \in \Omega$, all $\xi \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n} a_j(x, \xi) \xi_j \geq c_2 |\xi|^p - k_2(x).$$

**Remark 3.2.** Assumption $(A4)$ implies that for any $u \in V$,

$$\langle A(u), u\rangle \geq c_2 \|u\|_V^p - \int_\Omega k_2(x)dx. \tag{3.4}$$

Now we formulate particular cases when $(A3)$ and $(A4)$ are fulfilled. First observe that $(A1)$–$(A4)$ are satisfied in the simple case:

$$a_j(x, \xi) = \alpha_j(x, \xi_j), \quad j = 0, 1, ..., n \tag{3.5}$$

where the Carathéodory function $\alpha_j$ satisfies the following conditions for all $j$, a.a. $x \in \Omega$:

$$\xi_j \mapsto \alpha_j(x, \xi_j) \text{ is monotone nondecreasing,} \tag{3.6}$$

$$\beta_1 |\xi_j|^{p-1} \leq |\alpha_j(x, \xi_j)| \leq \beta_2 |\xi_j|^{p-1}, \quad \xi_j \in \mathbb{R} \tag{3.7}$$

with some positive constants $\beta_1, \beta_2$. Thus, by Theorem 2.14 there exists a solution $u \in V$ of equation (2.12) if (3.5)-(3.7) hold.

**Proposition 3.3.** *Assume that the functions $a_j$ satisfy $(A1)$, for a.e. $x \in \Omega$, the functions $\xi \mapsto a_j(x, \xi)$ are continuously differentiable and the matrix*

$$\left(\frac{\partial a_j(x, \xi)}{\partial \xi_k}\right)_{j,k=0}^n \quad \textit{is positive semidefinite.} \tag{3.8}$$

*Then $(A3)$ is fulfilled, thus $A$, defined by (3.1) is monotone.*

*Proof.* For arbitrary fixed $x \in \Omega$, $\xi, \xi^\star \in \mathbb{R}^{n+1}$ define function $h_j$ by

$$h_j(\tau) = a_j(x, \xi^\star + \tau(\xi - \xi^\star)), \quad \tau \in \mathbb{R}.$$

Then

$$h_j(1) - h_j(0) = \int_0^1 h_j'(\tau) d\tau, \text{ i.e.}$$

$$a_j(x, \xi) - a_j(x, \xi^\star) = \int_0^1 \sum_{k=0}^n \frac{\partial a_j}{\partial \xi_k}(x, \xi^\star + \tau(\xi - \xi^\star))(\xi_k - \xi_k^\star) d\tau,$$

hence by (3.8)

$$\sum_{j=0}^n [a_j(x, \xi) - a_j(x, \xi^\star)](\xi_j - \xi_j^\star) = \tag{3.9}$$

$$\int_0^1 \sum_{j,k=0}^n \frac{\partial a_j}{\partial \xi_k}(x, \xi^\star + \tau(\xi - \xi^\star))(\xi_k - \xi_k^\star)(\xi_j - \xi_j^\star) d\tau \geq 0.$$

$\square$

**Proposition 3.4.** *Assume that conditions of Proposition 3.3 are fulfilled such that for a.e. $x \in \Omega$, each $\xi, \eta \in \mathbb{R}^{n+1}$*

$$\sum_{j,k=0}^{n} \frac{\partial a_j}{\partial \xi_k}(x,\xi)\eta_j \eta_k \geq c_3 \sum_{j=0}^{n} |\xi_j|^{p-2}|\eta_j|^2 \tag{3.10}$$

*with $p \geq 2$ and some positive constant $c_3$. Then*

$$\sum_{j=0}^{n}[a_j(x,\xi) - a_j(x,\xi^\star)](\xi_j - \xi_j^\star) \geq \tilde{c}_3 \sum_{j=0}^{n} |\xi_j - \xi_j^\star|^p \tag{3.11}$$

*with some constant $\tilde{c}_3 > 0$.*

*Proof.* By (3.9), (3.10)

$$\sum_{j=0}^{n}[a_j(x,\xi) - a_j(x,\xi^\star)](\xi_j - \xi_j^\star) = \tag{3.12}$$

$$\int_0^1 \sum_{j,k=0}^{n} \frac{\partial a_j}{\partial \xi_k}(x, \xi^\star + \tau(\xi - \xi^\star))(\xi_k - \xi_k^\star)(\xi_j - \xi_j^\star)d\tau \geq$$

$$\int_0^1 c_3 \sum_{j=0}^{n} |\xi_j^\star + \tau(\xi_j - \xi_j^\star)|^{p-2}|\xi_j - \xi_j^\star|^2 d\tau.$$

Now we show that there is a constant $c_4 > 0$ (depending only on $p$) such that

$$\int_0^1 |\xi_j^\star + \tau(\xi_j - \xi_j^\star)|^{p-2}d\tau \geq c_4|\xi_j - \xi_j^\star|^{p-2}. \tag{3.13}$$

Clearly, for $\xi_j - \xi_j^\star = 0$ (3.13) holds. For $\xi_j - \xi_j^\star \neq 0$ we have

$$\int_0^1 |\xi_j^\star + \tau(\xi_j - \xi_j^\star)|^{p-2}d\tau = |\xi_j - \xi_j^\star|^{p-2} \int_0^1 |\xi_j^\star/(\xi_j - \xi_j^\star) + \tau|^{p-2}d\tau.$$

By using the notation $d = \xi_j^\star/(\xi_j - \xi_j^\star)$, we have to show that there is a constant $c_4 > 0$, not depending on $d$ such that

$$\int_0^1 |d + \tau|^{p-2}d\tau \geq c_4. \tag{3.14}$$

In the case $0 < -d < 1$

$$\int_0^1 |d+\tau|^{p-2}d\tau = \int_0^{-d}(-d-\tau)^{p-2}d\tau + \int_{-d}^1 (d+\tau)^{p-2}d\tau = \tag{3.15}$$

$$\frac{(-d)^{p-1} + (d+1)^{p-1}}{p-1} \geq \frac{1}{2^{p-2}(p-1)}$$

where we used inequality

$$(a+b)^s \le 2^{s-1}(a^s + b^s), \quad a, b \ge 0, s \ge 1.$$

In the case when $d \ge 0$ or $d \le -1$, $d+\tau$ has the same sign for all $\tau \in [0,1]$, thus

$$\int_0^1 |d+\tau|^{p-2} d\tau \ge \int_0^1 |\tau|^{p-2} d\tau = \frac{1}{p-1}. \tag{3.16}$$

Inequalities (3.15), (3.16) imply (3.14) and so we have shown (3.13). Consequently, from (3.12) we obtain

$$\sum_{j=0}^{n} [a_j(x,\xi) - a_j(x,\xi^\star)](\xi_j - \xi_j^\star) \ge c_3 c_4 \sum_{j=0}^{n} |\xi_j - \xi_j^\star|^p$$

which completes the proof of (3.11). □

From Proposition 3.4 immediately follows

**Theorem 3.5.** *Assume that the conditions of Proposition 3.4 and (A1), (A2) are fulfilled. Then operator $A$, defined by (3.1) has the property such that for all $u_1, u_2 \in V$*

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \ge c_5 \|u_1 - u_2\|_V^p \tag{3.17}$$

*with some positive constant $c_5$.*

**Remark 3.6.** If (3.17) is satisfied, the operator $A : V \to V^\star$ is uniformly monotone. (See the Definition 2.15.) Thus the solution of (2.12) depends continuously on $F$, in this case for the solutions of $A(u_1) = F_1$, $A(u_2) = F_2$ we have

$$\|u_1 - u_2\|_V \le \text{const}\|F_1 - F_2\|_{V^\star}^{1/(p-1)}.$$

Further, according to Remark 2.18, the sequence $(u_n)$, constructed by Galerkin's method, converges to the solution $u$ with respect to the norm of $V$.

In the case when $A$ is defined by (3.1) and (3.17) holds, $A$ is called strongly elliptic.

**Remark 3.7.** Clearly, (3.17) implies that $A$ is strictly monotone. Further, the assumptions of Theorem 3.5 imply that $A : V \to V^\star$ is coercive, too. Indeed,

$$\langle A(u), u \rangle = \langle A(u) - A(0), u \rangle + \langle A(0), u \rangle \ge c_5 \|u\|_V^p + \|A(0)\|_{V^\star} \|u\|_V$$

which implies that $A$ is coercive since $p > 1$.

**Example 3.8.** A typical example satisfying the conditions of Theorem 3.5 is

$$\triangle_p u + cu|u|^{p-2}, \quad c > 0 \text{ is a constant, } p \ge 2,$$

where $\triangle_p$ is the $p$-Laplacian operator, defined by

$$\triangle_p u = \sum_{j=1}^{n} D_j(|Du|^{p-2}Du). \tag{3.18}$$

In this case the functions $a_j$ are defined by

$$a_j(x,\zeta) = \xi_j|\zeta|^{p-2}, \quad j=1,\ldots,n, \quad a_0(x,\xi_o) = c\xi_0|\xi_0|^{p-2} \tag{3.19}$$

where we used the notation $\zeta = (\xi_1,\ldots,\xi_n)$ Now we show that the inequality (3.10) holds in this case. For $\zeta \neq 0$, $j=1,...,n$

$$\frac{\partial a_j(x,\zeta)}{\partial \xi_k} = (p-2)\xi_j\xi_k|\zeta|^{p-4} \text{ if } k \neq j,$$

$$\frac{\partial a_j(x,\zeta)}{\partial \xi_j} = (p-2)\xi_j^2|\zeta|^{p-4} + |\zeta|^{p-2}, \quad j=1,...,n \text{ and}$$

$$\frac{\partial a_0(x,\xi_0)}{\partial \xi_0} = c(p-1)|\xi_0|^{p-2},$$

hence

$$\sum_{j,k=0}^{n} \frac{\partial a_j(x,\zeta)}{\partial \xi_k}\eta_j\eta_k = (p-2)|\zeta|^{p-4}\sum_{j,k=1}^{n}\xi_j\xi_k\eta_j\eta_k +$$

$$|\zeta|^{p-2}\sum_{j=1}^{n}\eta_j^2 + c(p-1)|\xi_0|^{p-2}\eta_0^2 =$$

$$(p-2)|\zeta|^{p-4}\left[\sum_{j=1}^{n}\xi_j\eta_j\right]^2 + |\zeta|^{p-2}\sum_{j=1}^{n}\eta_j^2 + c(p-1)|\xi_0|^{p-2}\eta_0^2 \geq$$

$$\text{const}\sum_{j=0}^{n}|\xi_j|^{p-2}\eta_j^2.$$

Now consider operator $A$, defined by (3.1) with the functions (3.19). Clearly, $(A1)$, $(A2)$ are fulfilled and by Theorem 3.5 we have (3.17).

**Remark 3.9.** Consider the case $V = W_0^{1,p}(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^n$. Then the norm in $V$ is equivalent with the norm

$$\|u\|' = \left[\sum_{j=1}^{n}\int_\Omega |D_j u|^p dx\right]^{1/p}.$$

(For the particular case $p=2$ see, e.g. [67], for the general case see [1].) Therefore, conditions of Theorem 3.5 are fulfilled for $\triangle_p$, i.e. for $a_0 = 0$.

**Remark 3.10.** In Section 1 we have shown that if $u$ is a solution of (2.12) with the operator (3.1), $V = W_0^{1,p}(\Omega)$ then we may consider $u$ as a weak solution of the equation (1.4) with homogeneous Dirichlet boundary condition. The case of nonhomogeneous boundary condition $u|_{\partial\Omega} = \varphi$ can be reduced to a problem with 0 boundary condition for $\tilde{u} = u - u_0$ if there exits a function $u_0 \in W^{1,p}(\Omega)$ with the property $u_0|_{\partial\Omega} = \varphi$.

**Remark 3.11.** If $u$ is a solution of (2.12) with the operator (3.1) and $V = W^{1,p}(\Omega)$ then $u$ can be considered as a weak solution of (1.4) with the following homogeneous Neumann type boundary condition:

$$\sum_{j=1}^{n} a_j(x,u,Du)\nu_j|_{\partial\Omega} + hu|_{\partial\Omega} = g. \tag{3.20}$$

By using Gauss's theorem it is easy to show that a function $u \in C^2(\overline{\Omega})$ satisfies the boundary value problem (1.4), (3.20) (with sufficiently smooth functions $a_j$) if and only if $u$ is a solution of (2.12) with the operator $A$ (which is a modification of (3.1)):

$$\langle A(u), v\rangle = \sum_{j=1}^{n} \int_\Omega a_j(x,u,Du)D_j v dx + \int_\Omega a_0(x,u,Du)v dx + \int_{\partial\Omega} huv d\sigma,$$

$u, v \in V$,

$$\langle F, v\rangle = \int_\Omega fv + \int_{\partial\Omega} gv d\sigma$$

and $V = W^{1,p}(\Omega)$. Indeed, assuming that $u \in C^2(\overline{\Omega})$ satisfies (1.4), (3.20) (with sufficiently smooth functions $a_j$), multiplying (1.4) by $v \in C^1(\overline{\Omega})$ and integrating over $\Omega$, we obtain by (3.20)

$$\langle F, v\rangle = \int_\Omega fv + \int_{\partial\Omega} gv d\sigma = \tag{3.21}$$

$$-\sum_{j=1}^{n} \int_\Omega v D_j[a_j(x,u,Du)]dx + \int_\Omega v a_0(x,u,Du)dx + \int_{\partial\Omega} gv d\sigma =$$

$$-\int_{\partial\Omega} v \sum_{j=1}^{n} a_j(x,u,Du)\nu_j d\sigma + \int_\Omega \left[\sum_{j=1}^{n} a_j(x,u,Du)D_j v + a_0(x,u,Du)v\right] dx +$$

$$\int_{\partial\Omega} gv d\sigma = \langle A(u), v\rangle.$$

Further, when $u \in C^2(\overline{\Omega})$ satisfies $A(u) = F$, first apply

$$\langle A(u), v\rangle = \langle F, v\rangle \tag{3.22}$$

to $v \in C_0^1(\Omega)$. Then from (3.21) we obtain

$$\int_\Omega fvdx = \int_\Omega \left\{ -\sum_{j=1}^n D_j[a_j(x,u,Du)] + a_0(x,u,Du) \right\} vdx$$

which implies (1.4) since $C_0^1(\Omega)$ is dense in $L^2(\Omega)$. Then apply (3.22) to $v \in C^1(\overline{\Omega})$, by using (1.4), (3.21) we find

$$\int_{\partial\Omega} gvd\sigma = \int_{\partial\Omega} v \sum_{j=1}^n a_j(x,u,Du)\nu_j d\sigma + \int_{\partial\Omega} huvd\sigma$$

which implies (3.20) since the restrictions of functions $v \in C^1(\overline{\Omega})$ are dense in $L^2(\partial\Omega)$.

## Problems

1. Prove that for the functions (3.5), satisfying (3.6), (3.7), the assumptions $(A1)$–$(A4)$ are fulfilled.

2. Let $\alpha, \beta : \Omega \to \mathbb{R}$ be measurable functions satisfying

$$c_1 \le \alpha(x) \le c_2, \quad c_1 \le \beta(x) \le c_2, \quad x \in \Omega$$

with some positive constants $c_1, c_2$. By using Example 3.8, show that

$$a_j(x,\zeta) = \alpha(x)\xi_j|\zeta|^{p-2}, \quad j = 1,\dots,n, \quad \zeta = (\xi_1, \dots \xi_n) \in \mathbb{R}^n, \quad x \in \Omega,$$

$$a_0(x,\zeta) = \beta(x)\xi_0|\xi_0|^{p-2}, \quad \xi_0 \in \mathbb{R}, \quad x \in \Omega$$

satisfy the assumptions of Theorem 3.5.

3. Define the weak solution of the Dirichlet problem

$$-\sum_{j=1}^n D_j[a_j(x,u,Du)] + a_0(x,u,Du) = f$$

$$u|_{\partial\Omega} = \varphi$$

as a function $u \in W^{1,p}(\Omega)$ satisfying

$$\langle A(u), v\rangle = \sum_{j=1}^n \int_\Omega a_j(x,u,Du)D_j v + \int_\Omega a_0(x,u,Du)v = \langle F, v\rangle$$

$$\text{for all } v \in W_0^{1,p}(\Omega), \quad u|_{\partial\Omega} = \varphi$$

where $\langle F, v\rangle = \int_\Omega fv$ and $u|_{\partial\Omega}$ denotes the trace of $u \in W^{1,p}(\Omega)$ on the boundary $\partial\Omega$.

Show that (for "sufficiently good") functions $a_j$, a function $u \in C^2(\overline{\Omega})$ is a classical solution of the above Dirichlet problem if and only if it is a weak solution.

4. Prove that if the assumptions of Theorem 3.5 are fulfilled and there exists $u_0 \in W^{1,p}(\Omega)$ such that $u_0|_{\partial\Omega} = \varphi$ then for each $F \in [W_0^{1,p}(\Omega)]^\star$ there exists a unique weak solution of the Dirichlet problem (in Problem **3**) with nonhomogeneous boundary condition. (See Remark 3.10.)

# 4 Application of pseudomonotone operators

Here we shall formulate more general conditions than $(A3)$ (they are natural generalizations of ellipticity in the linear case) which will imply that the operator (3.1) is pseudomonotone. In the proof we shall apply the following two theorems.

**Theorem 4.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary, $1 < p < \infty$. Then $W^{1,p}(\Omega)$ is compactly imbedded into $L^p(\Omega)$.*

The exact formulation on smoothness of $\partial\Omega$ and the proof of the above theorem can be found in [1].

**Remark 4.2.** Later we shall apply the following statements, too. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary. Then $W^{1,p}(\Omega)$ is compactly imbedded into $W^{1-\delta,p}(\Omega)$ for arbitrary $0 < \delta \le 1$. Further, the trace operator $W^{1-\delta,p}(\Omega) \to L^p(\partial\Omega)$ is bounded if $0 \le \delta < 1 - 1/p$

**Theorem 4.3.** *(Vitali's theorem) Let $M \subset \mathbb{R}^n$ be a Lebesgue measurable set. Assume that the functions $f_k : M \to \mathbb{R}$ are Lebesgue integrable, further, for a.a. $x \in M$, $\lim_{k\to\infty} f_k(x)$ exists and is finite. The functions $f_k$ are equiintegrable in the following sense: for arbitrary $\varepsilon > 0$ there exist $\delta > 0$ and $S \subset M$ of finite measure such that for all $k \in \mathbb{N}$*

$$\int_H |f_k(x)|dx < \varepsilon \text{ if } \lambda(H) < \delta \text{ and } \int_{M\setminus S} |f_k(x)|dx < \varepsilon.$$

*Then*

$$\lim_{k\to\infty} \int_M f_k(x)dx = \int_M f(x)dx.$$

**Remark 4.4.** It is easy to show that if $(f_k) \to f$ in $L^1(M)$ then the functions $f_k$ are equiintegrable. Further, by Hölder's inequality one obtains: if $(|g_k|^p)$ is equiintegrable and $(h_k)$is bounded in $L^q(M)$ $(1 < p < \infty)$ then $(g_k h_k)$ is eqiintegrable.

First we formulate simple cases when Theorems 4.1 and 4.3 imply that operator (3.1) is pseudomonotone.

**Theorem 4.5.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\partial\Omega$ is sufficiently smooth and functions $a_j$, satisfying (A1), (A2) have the particular form*

$$a_j(x,\xi) = \tilde{a}_j(x,\zeta), \quad j = 1, ..., n \text{ where } \zeta = (\xi_1, ..., \xi_n),$$

$$a_0(x,\xi) = \tilde{a}_0(x,\xi_0)$$

*and instead of assumption (A3) we assume*

$$\sum_{j=1}^{n} [\tilde{a}_j(x,\zeta) - \tilde{a}_j(x,\zeta^\star)](\xi_j - \xi_j^\star) \geq 0. \tag{4.1}$$

*Then the (bounded) operator A (defined by (3.1)) is pseudomonotone.*

*Proof.* Assume that

$$(u_k) \to u \text{ weakly in } V \text{ and } \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0 \tag{4.2}$$

Since $(u_k)$ is bounded in $W^{1,p}(\Omega)$, by Theorem 4.1 there is a subsequence $(u_{k_l})$ of $(u_k)$ which converges to $u$ with respect to the norm of $L^p(\Omega)$ and a.e. in $\Omega$.

Define operator $B$ by

$$\langle B(u), v\rangle = \sum_{j=1}^{n} \int_\Omega \tilde{a}_j(x, Du) D_j v dx + \int_\Omega u|u|^{p-2} v dx.$$

Then (4.1) implies that $B : V \to V^\star$ is monotone and by $(A1)$, $(A2)$ $B$ is hemicontinuous and bounded. Consequently, from Proposition 2.5 it follows that $B$ is pseudomonotone. Further,

$$\langle B(u), v\rangle = \langle A(u), v\rangle + \int_\Omega [u|u|^{p-2} - \tilde{a}_0(x,u)] v dx. \tag{4.3}$$

Since

$$\lim_{l\to\infty} \|u_{k_l} - u\|_{L^p(\Omega)} = 0,$$

and by $(A2)$

$$\|u_{k_l}|u_{k_l}|^{p-2} - \tilde{a}_0(x, u_{k_l})\|_{L^q(\Omega)} \text{ is bounded,}$$

Hölder's inequality implies

$$\lim_{k\to\infty} \int_\Omega [u_{k_l}|u_{k_l}|^{p-2} - \tilde{a}_0(x,u_{k_l})](u_{k_l} - u) dx = 0. \tag{4.4}$$

Thus we obtain from (4.2)

$$\limsup_{l\to\infty} \langle B(u_{k_l}), u_{k_l} - u\rangle \leq 0. \tag{4.5}$$

Since $B$ is pseudomonotone, (4.2), (4.5) imply

$$\lim_{l\to\infty} \langle B(u_{k_l}), u_{k_l} - u\rangle = 0, \tag{4.6}$$

$$(B(u_{k_l})) \to B(u) \text{ weakly in } V^\star. \tag{4.7}$$

By (4.3), (4.4), (4.6)

$$\lim_{l\to\infty}\langle A(u_{k_l}), u_{k_l}-u\rangle = 0. \tag{4.8}$$

Finally, $(u_{k_l}) \to u$ a.e., so by $(A1)$

$$u_{k_l}|u_{k_l}|^{p-2} - \tilde{a}_0(x, u_{k_l}) \to u|u|^{p-2} - \tilde{a}_0(x,u) \text{ a.e. in } \Omega.$$

By using Hölder's inequality, one shows that for a fixed $v \in V$, the sequence of functions

$$[u_{k_l}|u_{k_l}|^{p-2} - \tilde{a}_0(x, u_{k_l})]v$$

is equiintegrable (the $L^q(\Omega)$ norm of the term in brackets is bounded). Thus by Theorem 4.3

$$\lim_{l\to\infty}\int_\Omega [u_{k_l}|u_{k_l}|^{p-2} - \tilde{a}_0(x, u_{k_l})]v dx = \int_\Omega [u|u|^{p-2} - \tilde{a}_0(x,u)]v dx.$$

and so from (4.7) we obtain that

$$(A(u_{k_l})) \to A(u) \text{ weakly in } V^\star. \tag{4.9}$$

(4.8), (4.9) hold for the sequence $(u_k)$, too. Because, assuming that it is not true, by using Cantor's trick, we get a contradiction. □

Now we formulate other conditions which imply that operator $A$ of the form (3.1) is pseudomonotone. Instead of $(A3)$ assume (by using the notation $\xi = (\eta,\zeta)$, $\eta = \xi_0$, $\zeta = (\xi_1, \dots, \xi_n)$)

$(\tilde{A}3)$ There exists a constant $\tilde{c}_2 > 0$ such that for a.a. $x \in \Omega$, all $\eta \in \mathbb{R}$, $\zeta, \zeta^\star \in \mathbb{R}^n$

$$\sum_{j=1}^n [a_j(x,\eta,\zeta) - a_j(x,\eta,\zeta^\star)](\xi_j - \xi_j^\star) \geq \tilde{c}_2|\zeta - \zeta^\star|^p.$$

**Theorem 4.6.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\partial\Omega$ is sufficiently smooth and $(A1)$, $(A2)$, $(\tilde{A}3)$ hold. Then operator $A$ of the form (3.1) is bounded and pseudomonotone.*

*Proof.* According to Proposition 3.1 $A$ is bounded. Now we show that $A$ is pseudomonotone. Assume that

$$(u_k) \to u \text{ weakly in } V \text{ and } \limsup_{k\to\infty}\langle A(u_k), u_k - u\rangle \leq 0. \tag{4.10}$$

Since $W^{1,p}(\Omega)$ is compactly imbedded into $L^p(\Omega)$ (for bounded $\Omega$ with sufficiently smooth boundary, see Theorem 4.1), there is a subsequence of $(u_k)$, again denoted by $(u_k)$, such that

$$(u_k) \to u \text{ in } L^p(\Omega) \text{ and a.e. in } \Omega \tag{4.11}$$

Since $(D_j u_k)$ is bounded in $L^p(\Omega)$, we may assume (on the subsequence) that

$$(D_j u_k) \to D_j u \text{ weakly in } L^p(\Omega), \quad j = 1, ..., n. \tag{4.12}$$

Further,

$$\langle A(u_k), u_k - u\rangle = \int_\Omega a_0(x, u_k, Du_k)(u_k - u)dx + \tag{4.13}$$

$$\sum_{j=1}^n \int_\Omega [a_j(x, u_k, Du_k) - a_j(x, u_k, Du)](D_j u_k - D_j u)dx +$$

$$\sum_{j=1}^n \int_\Omega a_j(x, u_k, Du)(D_j u_k - D_j u)dx.$$

The first term on the right-hand side of (4.13) tends to 0 by (4.11) and Hölder's inequality, because the multipliers of $(u_k - u)$ are bounded in $L^q(\Omega)$ (by $(A2)$). Further, the third term on the right-hand side converges to 0, too, by (4.12) and because (4.11), $(A1)$, $(A2)$ and Vitali's theorem (Theorem 4.3) imply that

$$a_j(x, u_k, Du) \to a_j(x, u, Du) \text{ in } L^q(\Omega).$$

Consequently, (4.10), (4.13) imply

$$\limsup_{k\to\infty} \sum_{j=1}^n \int_\Omega [a_j(x, u_k, Du_k) - a_j(x, u_k, Du)](D_j u_k - D_j u)dx \le 0. \tag{4.14}$$

From $(\tilde{A}3)$, (4.14) we obtain

$$\lim_{k\to\infty} \int_\Omega |Du_k - Du|^p dx = 0 \tag{4.15}$$

and (for a subsequence)

$$(Du_k) \to Du \text{ a.e. in } \Omega. \tag{4.16}$$

Therefore, by $(A1)$, $(A2)$, (4.11), (4.15), (4.16) and Vitali's theorem (Theorem 4.3)

$$a_j(x, u_k, Du_k) \to a_j(x, u, Du) \text{ in } L^q(\Omega), \quad j = 0, 1, ..., n.$$

Thus by Hölder's inequality

$$(A(u_k)) \to A(u) \text{ weakly in } V^\star. \tag{4.17}$$

Finally, from (4.11), (4.15) and $(A2)$ one gets

$$\lim_{k\to\infty} \langle A(u_k), u_k - u\rangle = 0. \tag{4.18}$$

Since (4.17), (4.18) hold for a subsequence of $(u_k)$, by using Cantor's trick, we obtain (4.17), (4.18) for the original sequence. $\square$

**Remark 4.7.** According to the proof of the above theorem operator $A$ belongs to the class $(S)_+$ and it is demicontinuous.

## Browder's theorem

The following more general theorem is due to F. Browder (see [14]). Instead of $(A3)$, $(A4)$ we assume that

$(A3')$ for a.a. $x \in \Omega$, all $\eta \in \mathbb{R}$; $\zeta, \zeta^\star \in \mathbb{R}^n$, $\zeta \neq \zeta^\star$

$$\sum_{j=1}^{n} [a_j(x, \eta, \zeta) - a_j(x, \eta, \zeta^\star)](\xi_j - \xi_j^\star) > 0$$

where we used the notations $\eta = \xi_0$, $\zeta = (\xi_1, ..., \xi_n)$.

**Remark 4.8.** In the linear case assumption $(A3')$ means ellipticity.

$(A4')$ There exist a constant $c_2 > 0$ and $k_2 \in L^1(\Omega)$ such that

$$\sum_{j=0}^{n} a_j(x, \eta, \zeta)\xi_j \geq c_2|\zeta|^p - k_2(x).$$

**Theorem 4.9.** *Assume (A1), (A2), (A3′), (A4′). Then the (bounded) operator $A$, defined by (3.1) with an arbitrary (possibly unbounded) domain $\Omega \subset \mathbb{R}^n$, is pseudomonotone.*

*Proof.* Assume (4.2), i.e.

$$(u_k) \to u \text{ weakly in } V \text{ and } \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0. \tag{4.19}$$

We have to show that

$$\lim_{k\to\infty} \langle A(u_k), u_k - u\rangle = 0 \text{ and } (A(u_k)) \to A(u) \text{ weakly in } V^\star. \tag{4.20}$$

We shall show that (4.20) holds for a suitable subsequence of $(u_k)$, by Cantor's trick this will imply (4.20) for $(u_k)$, too.

Assume that $(\Omega_m)$ is a sequence of bounded domains with sufficiently smooth boundary $\partial\Omega_m$ such that $\Omega_m \subset \Omega_{m+1}$ and $\Omega = \cup_{m=1}^{\infty} \Omega_m$. By Theorem 4.1 for arbitrary fixed $m$ there is a subsequence of $(u_k)$ which is convergent in $L^p(\Omega_m)$ and so a subsequence of this subsequence is a.e. convergent to $u$ in $\Omega_m$. By using a "diagonal process" one obtains a subsequence of $(u_k)$ which converges to $u$ a.e. in $\Omega$. For simplicity, we shall denote this subsequence also by $(u_k)$, so we have

$$(u_k) \to u \text{ a.e. in } \Omega. \tag{4.21}$$

The main part of the proof of our theorem is showing

$$(Du_k) \to Du \text{ a.e. in } \Omega. \tag{4.22}$$

Set

$$p_k(x) = \sum_{j=1}^{n} [a_j(x, u_k, Du_k) - a_j(x, u, Du)](D_j u_k - D_j u) + \tag{4.23}$$

$$[a_0(x,u_k,Du_k)-a_0(x,u,Du)](u_k-u),$$

then

$$\langle A(u_k)-A(u),u_k-u\rangle=\int_\Omega p_k(x)dx$$

and so by (4.19)

$$\limsup_{k\to\infty}\int_\Omega p_k(x)dx\le 0. \tag{4.24}$$

Due to (4.23) we have

$$p_k(x)=\sum_{j=1}^n a_j(x,u_k,Du_k)D_ju_k+a_0(x,u_k,Du_k)u_k-g_k(x) \tag{4.25}$$

where

$$g_k(x)=\left[\sum_{j=1}^n a_j(x,u,Du)(D_ju_k-D_ju)+a_0(x,u,Du)(u_k-u)\right]+ \tag{4.26}$$

$$\left[\sum_{j=1}^n a_j(x,u_k,Du_k)D_ju+a_0(x,u_k,Du_k)u\right].$$

By $(A2)$

$$|g_k(x)|\le c_4[|u|^{p-1}+|Du|^{p-1}+k_1(x)][|u_k|+|Du_k|+|u|+|Du|]+ \tag{4.27}$$

$$c_5[|u_k|^{p-1}+|Du_k|^{p-1}+k_1(x)][|u|+|Du|],$$

thus Hölder's inequality implies that the sequence $(g_k)$ is equiintegrable. (See Remark 4.4.) Further, by Young's inequality from (4.27) we obtain that for arbitrary $\varepsilon>0$ there exist a constant $c(\varepsilon)$ and a function $k_4\in L^1(\Omega)$ such that

$$|g_k(x)|\le\varepsilon|Du_k|^p+c(\varepsilon)[|u_k|^p+|u|^p+|Du|^p+k_4(x)]. \tag{4.28}$$

Choosing sufficiently small $\varepsilon>0$, one obtains from $(A4')$, (4.25), (4.28)

$$p_k(x)\ge c_2|Du_k|^p-k_2(x)-|g_k(x)|\ge \tag{4.29}$$

$$\frac{c_2}{2}|Du_k|^p-c_6[|u_k|^p+|u|^p+|Du|^p+k_5(x)]$$

with some constant $c_6$ and $k_5\in L^1(\Omega)$. Let

$$p_k^+(x)=\max\{p_k(x),0\},\quad p_k^-(x)=-\min\{p_k(x),0\},$$

then by (4.29)

$$0\le p_k^-(x)\le k_2(x)+|g_k(x)|$$

where the sequence on the right hand side is equiintegrable, hence the sequence

$$(p_k^-)_{k\in\mathbb{N}} \text{ is equiintegrable.} \tag{4.30}$$

Now we show that $(p_k^-)$ converges to 0 a.e. in $\Omega$. Indeed, $p_k$ can be written in the form

$$p_k(x) = q_k(x) + r_k(x) + s_k(x) \tag{4.31}$$

where

$$q_k(x) = \sum_{j=1}^{n} [a_j(x, u_k, Du_k) - a_j(x, u_k, Du)](D_j u_k - D_j u),$$

$$r_k(x) = \sum_{j=1}^{n} [a_j(x, u_k, Du) - a_j(x, u, Du)](D_j u_k - D_j u),$$

$$s_k(x) = [a_0(x, u_k, Du_k) - a_0(x, u, Du)](u_k - u).$$

Denote by $\chi_k$ the characteristic function of the set $\{x : p_k^-(x) > 0\}$ then

$$-p_k^- = \chi_k q_k + \chi_k r_k + \chi_k s_k. \tag{4.32}$$

By (4.29)

$$\frac{c_2}{2}|Du_k|^p \le c_6[|u_k|^p + |u|^p + |Du|^p + k_5(x)] \text{ if } p_k(x) < 0,$$

hence by (4.21) the sequence $(\chi_k Du_k)$ is bounded for a.a. fixed $x$. Thus by (4.21), $(A2)$

$$(\chi_k r_k) \to 0 \text{ a.e. and } (\chi_k s_k) \to 0 \text{ a.e.}$$

Since $\chi_k q_k \ge 0$ a.e., it follows from (4.32)

$$(p_k^-) \to 0 \text{ a.e.} \tag{4.33}$$

Thus by (4.30) and Vitali's theorem

$$\lim_{k\to\infty} \int_\Omega p_k^- \, dx = 0. \tag{4.34}$$

Since $0 \le p_k^+ = p_k + p_k^-$, from (4.24), (4.34) we obtain

$$\lim_{k\to\infty} \int_\Omega p_k^+ \, dx = 0. \tag{4.35}$$

From (4.34), (4.35) it follows $\lim_{k\to\infty} \int_\Omega p_k = 0$ and so by (4.23) we obtain the first part of (4.20):

$$\langle A(u_k), u_k - u\rangle = \langle A(u_k) - A(u), u_k - u\rangle + \langle A(u), u_k - u\rangle =$$

$$\int_\Omega p_k(x)dx + \langle A(u), u_k - u\rangle \to 0.$$

By (4.35)

$$(p_k^+) \to 0 \text{ a.e., for a subsequence}$$

(again denoted by $(p_k^+)$, for simplicity). Thus (4.33) implies that

$$(p_k) \to 0 \text{ a.e.} \tag{4.36}$$

Hence (4.29) implies that for a.a. fixed $x \in \Omega$ the sequence $(Du_k(x))$ is bounded.

Consider such a fixed $x \in \Omega$. Assuming that (4.22) is not valid, we have a subsequence of $(Du_k(x))$, (again denoted by $(Du_k(x))$, for simplicity), which converges to some $\zeta^\star \neq (Du)(x)$. Since

$$(u_k(x)) \to u(x), \quad (r_k(x)) \to 0, \quad (s_k(x)) \to 0,$$

we obtain that

$$0 = \lim_{k\to\infty} p_k(x) = \sum_{j=1}^n [a_j(x, u(x), \zeta^\star) - a_j(x, u(x), Du(x))](\zeta_j^\star - D_j u(x)).$$

Thus by $(A3')$ we obtain $\zeta^\star = Du(x)$ which contradicts to $\zeta^\star \neq (Du)(x)$. So we have shown (4.22).

Hence we obtain the second part of (4.20), by using Vitali's theorem: for arbitrary fixed $v \in V$

$$\langle A(u_k), v\rangle = \sum_{j=1}^n \int_\Omega a_j(x, u_k, Du_k) D_j v dx + \int_\Omega a_0(x, u_k, Du_k) v dx \to$$

$$\sum_{j=1}^n \int_\Omega a_j(x, u, Du) D_j v dx + \int_\Omega a_0(x, u, Du) v dx$$

because the sequence of integrands is equiintegrable by $(A2)$ and Hölder's inequality, further, the a.e. convergence follows from $(A1)$, (4.21), (4.22). □

**Remark 4.10.** According to the proof of the above theorem, $A$ belongs to the class $(S)_+$ if $\Omega$ is bounded and it is demicontinuous.

**Remark 4.11.** If instead of $(A4')$ we assume $(A4)$, we obtain that $A$ is coercive, too and we have existence of solutions for arbitrary $F \in V^\star$. In the particular case when $\Omega$ is bounded and $V = W_0^{1,p}(\Omega)$, $(A4')$ implies that $A$ is coercive (see Remark 3.9).

**Remark 4.12.** F.E. Browder proved in [14] the following generalization of Theorem 4.9. Let $V \subset W^{m,p}(\Omega)$ be a closed linear subspace ($m \geq 1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ arbitrary, possibly unbounded domain) where $W^{m,p}(\Omega)$ denotes the Sobolev space of (real valued) measurable functions $u : \Omega \to \mathbb{R}$ with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left[\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u|^p dx\right]^{1/p},$$

$D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}$, $D_j = \partial/\partial x_j$. (For the detailed investigation of Sobolev spaces see, e.g., [1].) Define operator $A : V \to V^\star$ by the formula

$$\langle A(u), v\rangle = \sum_{|\alpha|\le m} \int_\Omega a_\alpha(x, u, ..., D^\beta u, ...)D^\alpha v dx \tag{4.37}$$

where $|\beta| \le m$ and functions $a_\alpha$ (depending on a multiindex $\alpha$) satisfy the natural generalizations of $(A1)$, $(A2)$, $(A3')$, $(A4')$. Then $A$ is pseudomonotone.

A similar generalization of Theorem 4.6 can be formulated and proved for higher order nonlinear elliptic equations.

The proofs of the generalizations are similar to that of Theorems 4.9, 4.6, respectively.

**Example 4.13.** A simple example satisfying the assumptions of Theorem 4.4, where $A$ is coercive is:

$$-\triangle_p u + a_0(x, u, Du) = F$$

where the function $a_0$ satisfies $(A1)$, $(A2)$ and

$$a_0(x, \xi)\xi_0 \ge c_2|\xi_0|^p \tag{4.38}$$

with some constant $c_2 > 0$. If $\Omega$ is bounded and $V = W_0^{1,p}(\Omega)$, instead of (4.38) it is sufficient to assume $a_0(x, \xi)\xi_0 \ge 0$ (see Remark 3.9).

## Nonlinear elliptic functional equations

Now we apply the theory of pseudomonotone operators to nonlinear elliptic functional equations with nonlinear and "non-local" third boundary conditions. Let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace ($1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ a bounded domain with sufficiently smooth boundary).

**Definition 4.14.** *Define operator $A$ by*

$$\langle A(u), v\rangle = \int_\Omega \left[\sum_{j=1}^n a_j(x, u(x), Du(x); u)D_j v(x) + a_0(x, u(x), Du(x); u)v(x)\right] dx+ \tag{4.39}$$

$$\int_{\partial\Omega} h(x; u)v d\sigma, \quad u, v \in V.$$

Assume that the following conditions are fulfilled.

$(A1^\star)$ The functions $a_j : \Omega \times \mathbb{R}^{n+1} \times V \to \mathbb{R}$ ($j = 0, 1, ..., n$) satisfy the Carathéodory conditions for arbitrary fixed $u \in V$ and $h : \partial\Omega \times V \to \mathbb{R}$ is measurable for each fixed $u \in V$.

($A2^\star$) There exist bounded (nonlinear) operators $g_1 : V \to \mathbb{R}^+$ and $k_1 : V \to L^q(\Omega)$ such that

$$|a_j(x,\eta,\zeta;u)| \le g_1(u)[1+|\eta|^{p-1}+|\zeta|^{p-1}] + [k_1(u)](x), \quad j=0,1,...,n$$

for a.e. $x \in \Omega$, each $(\eta,\zeta) \in \mathbb{R}^{n+1}$, $u \in V$.

($A3^\star$) The inequality

$$\sum_{j=1}^{n}[a_j(x,\eta,\zeta;u) - a_j(x,\eta,\zeta^\star;u)](\xi_j - \xi_j^\star) \ge g_2(u)|\zeta-\zeta^\star|^p$$

holds where

$$g_2(u) \ge c^\star \left[1+\|u\|_V\right]^{-\sigma^\star} \tag{4.40}$$

and the constants $c^\star, \sigma^\star$ satisfy $c^\star > 0$, $0 \le \sigma^\star < p-1$.

($A4^\star$) The inequality

$$\sum_{j=1}^{n} a_j(x,\eta,\zeta;u)\xi_j \ge g_2(u)[1+|\eta|^p+|\zeta|^p] - [k_2(u)](x) \tag{4.41}$$

holds where $k_2(u)$ and $h(x;u)$ satisfy with some positive $\sigma < p - \sigma^\star$, $\lambda_1 < p-1-\sigma^\star$

$$\|k_2(u)\|_{L^1(\Omega)} \le \text{const}\left[1+\|u\|_V\right]^{\sigma}, \quad u \in V, \tag{4.42}$$

$$\|h(x;u)\|_{L^q(\partial\Omega)} \le \text{const}\left[1+\|u\|_V\right]^{\lambda_1}, \quad u \in V. \tag{4.43}$$

(In the case $V = W_0^{1,p}(\Omega)$ $h$ is considered to be identically 0.)

($A5^\star$) There exists $\delta > 0$ satisfying $\delta < 1 - 1/p$ such that if $(u_k) \to u$ weakly in $V$ and strongly in $W^{1-\delta,p}(\Omega)$, $(\eta^k) \to \eta$ in $\mathbb{R}$, $(\zeta^k) \to \zeta$ in $\mathbb{R}^n$ then for a.a. $x \in \Omega$, $j = 0,1,\dots,n$

$$\lim_{k\to\infty} a_j(x,\eta^k,\zeta^k;u_k) = a_j(x,\eta,\zeta;u)$$

for a subsequence and for a.a. $x \in \Omega$

$$\lim_{k\to\infty} h(x;u_k) = h(x;u)$$

for a suitable subsequence.

**Theorem 4.15.** *Assume ($A1^\star$) – ($A5^\star$). Then $A : V \to V^\star$ is bounded, pseudomonotone and coercive. Thus for any $F \in V^\star$ there exists $u \in V$ satisfying $A(u) = F$.*

*Proof.* Clearly, ($A1^\star$), ($A2^\star$) and (4.43) imply that $A$ is bounded, because the trace operator $W^{1-\delta,p}(\Omega) \to L^p(\partial\Omega)$ is bounded by $\delta + 1/p < 1$ (see [1]) and so by Hölder's inequality

$$\left|\int_{\partial\Omega} h(x;u)v(x)d\sigma\right| \le \left[\int_{\partial\Omega}|h(x;u)|^q d\sigma\right]^{1/q}\left[\int_{\partial\Omega}|v(x)|^p d\sigma\right]^{1/p} \le \tag{4.44}$$

$$\text{const}[1+\|u\|_V]^{\lambda_1}\|v\|_{W^{1-\delta,p}(\Omega)} \le \text{const}[1+\|u\|_V]^{\lambda_1}\|v\|_{W^{1,p}(\Omega)}.$$

Assumption $(A4^\star)$ implies that $A$ is coercive because by (4.44)

$$\langle A(u_k),u_k\rangle \ge c^\star[1+\|u_k\|_V]^{p-\sigma^\star} - \text{const}[1+\|u_k\|_V]^{\sigma} -$$

$$\text{const}[1+\|u_k\|_V]^{\lambda_1+1} \to +\infty$$

as $\|u_k\|_V \to \infty$ since $p-\sigma^\star > \sigma$, $p-\sigma^\star > \lambda_1+1$, $p-\sigma^\star > 1$.

Now we show (similarly to the proof of Theorem 4.6) that $A$ is pseudomonotone. Assume that

$$(u_k) \to u \text{ weakly in } V \text{ and } \limsup_{k\to\infty}\langle A(u_k),u_k-u\rangle \le 0. \tag{4.45}$$

Since $W^{1,p}(\Omega)$ is compactly imbedded into $W^{1-\delta,p}(\Omega)$ (for bounded $\Omega$ with sufficiently smooth boundary, see [1]), there is a subsequence of $(u_k)$, again denoted by $(u_k)$, for simplicity, such that

$$(u_k) \to u \text{ in } W^{1-\delta,p}(\Omega) \text{ and a.e. in } \Omega \tag{4.46}$$

and by $(A5^\star)$

$$h(x;u_k) \to h(x;u) \text{ for a.e. } x \in \partial\Omega. \tag{4.47}$$

Since $(D_j u_k)$ is bounded in $L^p(\Omega)$, we may assume (on the subsequence) that

$$(D_j u_k) \to D_j u \text{ weakly in } L^p(\Omega), \quad j=1,...,n. \tag{4.48}$$

Further,

$$\langle A(u_k),u_k-u\rangle = \int_\Omega a_0(x,u_k,Du_k;u_k)(u_k-u)dx + \tag{4.49}$$

$$\sum_{j=1}^n \int_\Omega [a_j(x,u_k,Du_k;u_k) - a_j(x,u_k,Du;u_k)](D_j u_k - D_j u)dx +$$

$$\sum_{j=1}^n \int_\Omega a_j(x,u_k,Du;u_k)(D_j u_k - D_j u)dx + \int_{\partial\Omega} h(x;u_k)(u_k-u)d\sigma.$$

The first and the fourth terms on the right hand side of (4.49) tend to 0 by (4.46) and Hölder's inequality, because the multipliers of $(u_k-u)$ are bounded in $L^q(\Omega)$ and $L^q(\partial\Omega)$, respectively (by $(A2^\star)$ and (4.43)), and the trace operator $W^{1-\delta,p}(\Omega) \to L^p(\partial\Omega)$ is continuous. Further, the third term on the right hand side converges to 0, too, by (4.48) because (4.45), (4.46), $(A1^\star)$, $(A2^\star)$, $(A5^\star)$ and Vitali's theorem (Theorem 4.3) imply that

$$a_j(x,u_k,Du;u_k) \to a_j(x,u,Du;u) \text{ in } L^q(\Omega).$$

Consequently, (4.45), (4.49) imply

$$\limsup_{k\to\infty} \sum_{j=1}^n \int_\Omega [a_j(x,u_k,Du_k;u_k) - a_j(x,u_k,Du;u_k)](D_j u_k - D_j u)dx \le 0. \tag{4.50}$$

Since $(u_k)$ is bounded in $V$, from $(A3^\star)$, (4.50) we obtain

$$\lim_{k\to\infty} \int_\Omega |Du_k - Du|^p dx = 0 \tag{4.51}$$

and (for a subsequence)

$$(Du_k) \to Du \text{ a.e. in } \Omega. \tag{4.52}$$

Therefore, by $(A1^\star)$, $(A2^\star)$, $(A5^\star)$, (4.45), (4.46), (4.52) and Vitali's theorem (Theorem 4.3)

$$a_j(x, u_k, Du_k; u_k) \to a_j(x, u, Du; u) \text{ in } L^q(\Omega), \quad j = 0, 1, ..., n.$$

Thus by Hölder's inequality, (4.44), (4.47) and Vitali's theorem

$$(A(u_k)) \to A(u) \text{ weakly in } V^\star. \tag{4.53}$$

Finally, from (4.44), (4.46), (4.51) and $(A2^\star)$ one gets

$$\lim_{k\to\infty} \langle A(u_k), u_k - u\rangle = 0. \tag{4.54}$$

Since (4.53), (4.54) hold for a subsequence of $(u_k)$, by using Cantor's trick, we obtain (4.53), (4.54) for the original sequence.

So we have proved that $A$ is bounded, pseudomonotone and coercive, thus Theorem 2.12 implies Theorem 4.15. $\square$

**Remark 4.16.** The solution $u$ of the equation $A(u) = F$ with operator (4.39) can be considered as weak solution of the equation

$$-\sum_{j=1}^n D_j[a_j(x, u, Du; u)] + a_0(x, u, Du; u) = f \tag{4.55}$$

with the "non-local" third boundary condition

$$\sum_{j=1}^n a_j(x, u, Du; u)\nu_j + h(x; u) = 0 \text{ on } \partial\Omega. \tag{4.56}$$

Indeed, by using Gauss's theorem, it is easy to show that a function $u \in C^2(\overline{\Omega})$ satisfies the boundary value problem (4.55), (4.56) (with sufficiently smooth $a_j(x, u, Du; u)$ if and only if $u$ is a solution of $A(u) = F$ with operator (4.39), $\langle F, v\rangle = \int_\Omega fv dx$ and $V = W^{1,p}(\Omega)$. (See Remark 3.11.)

By using the Rellich-Kondrashov compact imbedding theorem, one is able to prove an existence theorem on equation $A(u) = F$ for the operator (4.39) with a more general growth condition than $(A2^\star)$. The Rellich-Kondrashov theorem with respect to the space $W^{1,p}(\Omega)$ says (see, e.g., [1]):

**Theorem 4.17.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with "sufficiently good" boundary ($\Omega$ has the "cone property", see [1]);*

$$1 \leq p_1 < \frac{np}{n-p} \text{ if } p < n, \quad 1 \leq p_1 \text{ arbitrary if } p = n,$$

$$p_1 = \infty \text{ if } p > n.$$

*Then $W^{1,p}(\Omega)$ is compactly imbedded into $L^{p_1}(\Omega)$.*

Now instead of ($A2^\star$) assume

($A2$") There exist bounded (nonlinear) operators $g_1 : V \to \mathbb{R}^+$ and $k_1 : V \to L^q(\Omega)$ such that for $j = 1, ..., n$

$$|a_j(x, \eta, \zeta; u)| \leq g_1(u)[1 + |\eta|^{p_1/q} + |\zeta|^{p-1}] + [k_1(u)](x) \text{ and}$$

$$|a_0(x, \eta, \zeta; u)| \leq g_1(u)[1 + |\eta|^{p_1/q_1} + |\zeta|^{p/q_1}] + [\tilde{k}_1(u)](x)$$

where $p_1$ is defined in Theorem 4.17, $1/p_1 + 1/q_1 = 1$ and $\tilde{k}_1 : V \to L^{q_1}(\Omega)$ is a bounded operator.

**Theorem 4.18.** *Assume ($A1^\star$), ($A2$"), ($A3^\star$)–($A5^\star$). Then the operator, defined by (4.39) is bounded, pseudomonotone and coercive. Thus for any $F \in V^\star$ there exists $u \in V$ satisfying $A(u) = F$.*

The proof is similar to that of Theorem 4.15. Applying Hölder's inequality also in $L^{p_1}(\Omega)$, $L^{q_1}(\Omega)$, we obtain by Theorem 4.17 that $A : V \to V^\star$ is bounded. Further, one proves that the first and third terms on the right hand side of (4.49) converge to 0, by using Hölder's inequality also in $L^{p_1}(\Omega)$, $L^{q_1}(\Omega)$ and Vitali's theorem. Finally, proving (4.53), we apply Vitali's theorem and Hölder's inequality also in $L^{p_1}(\Omega)$, $L^{q_1}(\Omega)$.

**Example 4.19.** Now we formulate examples satisfying ($A1^\star$)–($A5^\star$) (i.e. assumptions of Theorem 4.15). Set

$$a_j(x, \eta, \zeta; u) = b(x, [H(u)](x))\xi_i|\zeta|^{p-2}, \quad j = 1, ..., n,$$

$$a_0(x, \eta, \zeta; u) = b_0(x, [H_0(u)](x))\eta|\eta|^{p-2} + \hat{b}_0(x, [F_0(u)](x))\hat{\alpha}_0(x, \eta, \zeta),$$

$$h(x; u) = \beta(x, [G(u)](x))$$

where $b, b_0, \hat{b}_0, \hat{\alpha}_0$ $\beta$ are Carathéodory functions and they satisfy

$$b(x, \theta) \geq \frac{c_2}{1 + |\theta|^{\sigma^\star}}, \quad b_0(x, \theta) \geq \frac{c_2}{1 + |\theta|^{\sigma^\star}}$$

with some constants $c_2 > 0$, $0 \leq \sigma^\star < p - 1$,

$$\hat{b}_0(x, \theta) \leq 1 + |\theta|^{p-1-\varrho^\star} \text{ with } 0 < \varrho^\star < p - 1$$

$$|\hat{\alpha}_0(x, \eta, \zeta)| \leq c_1[1 + |\eta|^{\hat{\varrho}} + |\zeta|^{\hat{\varrho}}], \quad 0 \leq \hat{\varrho}, \quad \sigma^\star + \hat{\varrho} < \varrho^\star,$$

$$|\beta(x, \theta)| \leq c_1[1 + |\theta|^{\lambda_1}], \quad 0 < \lambda_1 < p - 1 - \sigma^\star.$$

Finally,

$$H, H_0 : W^{1-\delta,p}(\Omega) \to C(\overline{\Omega}), \quad F_0 : W^{1-\delta,p}(\Omega) \to L^p(\Omega), \quad G : L^p(\partial\Omega) \to L^p(\partial\Omega)$$

are linear continuous operators. Clearly, assumptions $(A1^\star)$ – $(A3^\star)$, $(A5^\star)$ are fulfilled, we have to show only the estimate $(A2^\star)$ for the second term in $a_0(x, \eta, \zeta; u)$. By Young's inequality

$$|\hat{b}_0(x, [F_0(u)](x))\hat{\alpha}_0(x, \eta, \zeta)| \leq [1 + |F_0(u)|^{p-1-\varrho^\star}]c_1[1 + |\eta|^{\hat{\varrho}} + |\zeta|^{\hat{\varrho}}] \leq$$

$$\text{const}[1 + |\eta|^{\hat{\varrho}} + |\zeta|^{\hat{\varrho}}]^{p_1} + \text{const}[1 + |F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

where

$$p_1 = \frac{p-1}{\hat{\varrho}} > 1, \quad q_1 = \frac{p_1}{p_1 - 1} = \frac{p-1}{p-1-\hat{\varrho}}.$$

Consequently, we obtain for this term $(A2^\star)$ with

$$k_1(u) = \text{const}[1 + |F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

since by Hölder's inequality we have for this term

$$\int_\Omega |k_1(u)|^q = \text{const} \int_\Omega [1 + |F_0(u)|^{(p-1-\varrho^\star)q_1 q}] \leq$$

$$\text{const}\left[1 + \int_\Omega |F_0(u)|^p\right]^{\mu/p} \leq \text{const}\,[1 + \|u\|_V^\mu]$$

where

$$\mu = (p-1-\varrho^\star)q_1 q = \frac{p-1-\varrho^\star}{p-1-\hat{\varrho}}p < p.$$

Now we prove that $(A4^\star)$ holds. Clearly, for our example we have in (4.40)

$$g_2(u) = \min\left\{\frac{\text{const}}{1 + \|H(u)\|^{\sigma^\star}_{C(\overline{\Omega})}}, \frac{\text{const}}{1 + \|H_0(u)\|^{\sigma^\star}_{C(\overline{\Omega})}}\right\} \geq \text{const}[1 + \|u\|_V]^{-\sigma^\star}.$$

Further, by Young's inequality

$$|\hat{b}_0(x, [F_0(u)](x))\hat{\alpha}_0(x, \eta, \zeta)\eta| \leq [1 + |F_0(u)|^{p-1-\varrho^\star}]\text{const}[1 + |\eta|^{\hat{\varrho}+1} + |\zeta|^{\hat{\varrho}+1}] \leq$$

$$\frac{\varepsilon^p}{p}[1 + |\eta|^{p-\sigma^\star} + |\zeta|^{p-\sigma^\star}] + C(\varepsilon)[1 + |F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

for any $\varepsilon > 0$ (because $\hat{\rho} + 1 < p - \sigma^\star$) where

$$p_1 = \frac{p-\sigma^\star}{\hat{\varrho}+1} > 1, \quad q_1 = \frac{p_1}{p_1 - 1} = \frac{p-\sigma^\star}{p-\sigma^\star-\hat{\varrho}-1}$$

and $C(\varepsilon)$ is a constant, depending on $\varepsilon$. Choosing sufficiently small $\varepsilon > 0$, we obtain $(A4^\star)$ with

$$[k_2(u)](x) = C(\varepsilon)[1 + |F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

since

$$\|k_2(u)\|_{L^1(\Omega)} = \text{const} \int_\Omega [1 + |F_0(u)|^{(p-1-\rho^\star)q_1}] =$$

$$\text{const} \int_\Omega [1 + |F_0(u)|^{(p-\sigma^\star)\lambda}] \le \text{const} \left[1 + \|F_0(u)\|_{L^p(\Omega)}^{(p-\sigma^\star)\lambda}\right] \le$$

$$\text{const} \left[1 + \|u\|_{W^{1-\delta,p}(\Omega)}^{(p-\sigma^\star)\lambda}\right] \le \text{const}[1 + \|u\|_V]^\sigma$$

with $\sigma = (p - \sigma^\star)\lambda$ where

$$\lambda = \frac{p - 1 - \varrho^\star}{p - 1 - \sigma^\star - \hat{\varrho}} < 1$$

because $\sigma^\star + \hat{\varrho} < \varrho^\star$ and thus $(p - \sigma^\star)\lambda < p - \sigma^\star$.

If functions $b, b_0$ are between two positive constants then, clearly, $(A1^\star)$ – $(A5^\star)$ are fulfilled when

$$H, H_0 : W^{1-\delta,p}(\Omega) \to L^p(\Omega)$$

are continuous linear operators (as $F_0$). So in this case $[H(u)](x)$, $[H_0(u)](x)$ (and $[F_0(u)](x)$) may have also e.g. the forms

$$\int_\Omega d(x,\xi)u(\xi)d\xi \text{ where } \int_\Omega \left[\int_\Omega |d(x,\xi)|^q d\xi\right]^{p/q} dx < \infty$$

or $u(\chi(x))$ where $\chi, \chi^{-1} : \overline{\Omega} \to \overline{\Omega}$ are continuously differentiable.

Finally,

$$\int_{\partial\Omega} |h(x;u)|^q d\sigma \le \int_{\partial\Omega} |\beta(x, [G(u)](x))|^q d\sigma \le \text{const} \int_{\partial\Omega} [1 + |G(u)|^{\lambda_1 q}] d\sigma \le$$

$$\text{const} \left[\int_{\partial\Omega} (1 + |G(u)|^p) d\sigma\right]^{\lambda_1 q/p} \le \text{const} \left[1 + \|u\|_{L^p(\partial\Omega)}^{\lambda_1 q}\right] \le \text{const} \left[1 + \|u\|_V^{\lambda_1 q}\right]$$

which implies (4.43).

## Problems

1. Prove Remark 4.7.

2. Show that the Example 4.13 satisfies the assumptions of Theorem 4.6.

3. Prove Theorem 4.18.

4. Assume that the functions $a_j$ satisfy the conditions $(A1)$, $(A2)$, $(\tilde{A}3)$, $(A4)$ and there exists $u_0 \in W^{1,p}(\Omega)$ such that $u_0|_{\partial\Omega} = \varphi$. Prove that then for each $F \in [W_0^{1,p}(\Omega)]^\star$ there exists a weak solution of the Dirichlet problem with nonhomogeneous boundary condition, considered in Problem **3** in Section 3. (See Remark 3.10.)

5. Let $V$ be a closed linear subspace of $W^{m,p}(\Omega)$ ($m \geq 1, 1 < p < \infty$) and consider the operator (4.37). Denote by $N$ the number of multiindices $\beta = (\beta_1, \dots \beta_n)$ satisfying $|\beta| = \sum_{j=1}^{n} \beta_j \leq m$. Assume that the functions $a_\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory conditions, i.e.

$$x \mapsto a_\alpha(x, \xi) \text{ is measurable for each } \xi \in \mathbb{R}^N,$$

$$\xi \mapsto a_\alpha(x, \xi) \text{ is continuous for a.a. } x \in \Omega.$$

   Further, there exist a constant $c_1 > 0$ and a function $k_1 \in L^q(\Omega)$ such that

$$|a_\alpha(x, \xi)| \leq c_1 |\xi|^{p-1} + k_1(x), \quad \xi \in \mathbb{R}^N, \text{ a.a. } x \in \Omega$$

   Prove that then the operator (4.37) is bounded.

6. Consider the operator (4.37) satisfying the assumptions of Problem 5. Denote by $M$ the number of multiindices $\beta$ satisfying $|\beta| = m$. Assume that there exists a positive constant $c_2$ such that

$$\sum_{|\alpha|=m} [a_\alpha(x, \eta, \zeta) - a_\alpha(x, \eta, \zeta^\star)](\zeta_\alpha - \zeta^\star_\alpha) \geq c_2 |\zeta - \zeta^\star|^p \tag{4.57}$$

   for a.a. $x \in \Omega$, all $\zeta, \zeta^\star \in \mathbb{R}^M$, $\eta \in \mathbb{R}^{N-M}$. By using the arguments of the proof of Theorem 4.6, prove that the bounded operator $A : V \to V^\star$ is pseudomonotone.

7. By using Proposition 3.4 formulate conditions, which imply the inequality (4.57).

8. Formulate assumptions on functions $a_\alpha$ which imply that the operator A defined by (4.37) is coercive. Show that the solution of $A(u) = F$ can be considered as a weak solution of the equation

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha [a_\alpha(x, u, \dots, D^\beta u, \dots)] = f \text{ in } \Omega$$

   with homogeneous Dirichlet conditions on $\partial\Omega$ if $V = W_0^{m,p}(\Omega)$ and with homogeneous Neumann conditions if $V = W^{m,p}(\Omega)$.

9. Let $V$ be a closed linear subspace of $W^{m,p}(\Omega)$ ($m \geq 1, p \geq 2$) and define the operator $A : V \to V^\star$ by

$$\langle A(u), v \rangle = \sum_{|\alpha| \leq m} \int_\Omega (D^\alpha u) |D^\alpha u|^{p-2} D^\alpha v dx, \quad u, v \in V.$$

   Prove that $A$ is bounded, demicontinuous, uniformly monotone, satisfies (3.17) and, consequently, $A$ is coercive.

10. Consider the operator (4.37) with $p \geq 2$, $V = W_0^{m,p}(\Omega)$. By using the notations of Problem 5, assume that the functions $a_\alpha$ have the form

$$a_\alpha(x, \xi) = \xi_\alpha |\xi_\alpha|^{p-2}, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N \text{ if } |\alpha| = m$$

and for $|\alpha| < m$ the functions $a_\alpha$ satisfy the assumptions of Problem 5, further,

$$a_\alpha(x, \xi)\xi_\alpha \geq 0, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N.$$

By using the fact that in $W_0^{m,p}(\Omega)$

$$\|u\|' = \left[ \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p \right]^{1/p}$$

is equivalent to the original norm, show that $A : V \to V^\star$ is bounded, pseudomonotone and coercive.

# 5 Nonlinear elliptic variational inequalities

## Preliminaries

In order to explain the importance of elliptic variational inequalities , first consider the weak solution of the linear elliptic equation (1.1) with homogeneous Dirichlet boundary condition, i.e. a function $u \in H_0^1(\Omega)$ satisfying for all $v \in H_0^1(\Omega)$

$$\langle Au, v \rangle = \sum_{j,k=1}^n \int_\Omega a_{jk}(D_k u)(D_j v)dx + \int_\Omega cuvdx = \int_\Omega fvdx = \langle F, v \rangle. \quad (5.1)$$

It is well-known (see, e.g., [67]) that if $c \geq 0$, $a_{jk} = a_{kj} \in L^\infty(\Omega)$ satisfy the uniform ellipticity condition then the unique solution $u \in H_0^1(\Omega)$ of (5.1) is the unique function $u = u^\star \in H_0^1(\Omega)$ which minimizes the quadratic functional

$$E(u) = \langle Au, u \rangle - 2\langle F, u \rangle = \quad (5.2)$$

$$\sum_{j,k=1}^n \int_\Omega a_{jk}(D_k u)(D_j u)dx + \int_\Omega cu^2 dx - 2\int_\Omega fudx.$$

(Here $A : V \to V^\star$ is a linear operator, $V = H_0^1(\Omega)$.)

Similarly, the weak solution of the Neumann problem with homogeneous boundary condition, i.e. the solution $u \in H^1(\Omega)$ of (5.1) for all $v \in H^1(\Omega)$, is the unique $u = u^\star \in H^1(\Omega)$ where $E$ attains its minimum in $H^1(\Omega)$.

By using similar arguments as in [67], one can show the following generalization of the above statements.

**Theorem 5.1.** *Let $K$ be a closed convex subset of the real Hilbert space $V$, $A : V \to V^\star$ be a bounded, strictly positive selfadjoint linear operator and $F \in V^\star$. Then the quadratic functional*

$$E(u) = \langle Au, u\rangle - 2\langle F, u\rangle \tag{5.3}$$

*attains its minimum in $K$ at $u = u^\star \in K$ where $u = u^\star$ is the unique solution of the "variational inequality"*

$$\langle Au, v - u\rangle \geq \langle F, v - u\rangle \text{ for all } v \in K. \tag{5.4}$$

*Proof.* The functional $E$ is bounded from below:

$$E(u) \geq c_0^2\|u\|_V^2 - 2\|F\|_{V^\star}\|u\|_V \geq \left[c_0\|u\|_V - \frac{\|F\|_{V^\star}}{c_0}\right]^2 - \frac{\|F\|_{V^\star}^2}{c_0^2} \geq$$

$$-\frac{\|F\|_{V^\star}^2}{c_0^2}$$

Let $(u_j)$ be a sequence such that

$$u_j \in K, \quad \lim_{j\to\infty} E(u_j) = \inf_K E = d. \tag{5.5}$$

As in [67], one can show that $(u_j)$ is a Cauchy sequence in $V$. Indeed, by using the parallelogram equality and (5.5), we obtain that for arbitrary $\varepsilon > 0$ there exists $j_0$ such that $j, l > j_0$ implies

$$\|u_j - u_l\|^2 = 2[\|u_j\|^2 + \|u_l\|^2] - \|u_j + u_l\|^2 = 2[E(u_j) + E(u_l)] - 4E\left(\frac{u_j + u_l}{2}\right) \leq$$

$$2[(d+\varepsilon) + (d+\varepsilon)] - 4d = 4\varepsilon.$$

Thus there is $u^\star \in V$ such that $\lim(u_j) = u^\star$. Since $u_j \in K$ and $K$ is closed, we obtain $u^\star \in K$. The continuity of $E$ implies

$$E(u^\star) = \lim_{j\to\infty} E(u_j) = \inf_K E. \tag{5.6}$$

The solution of (5.6) is unique, because if $E(\tilde{u}) = \inf_K E$ then

$$u^\star, \tilde{u}, u^\star, \tilde{u}, \ldots$$

must be a Cauchy sequence according to the above argument.

Now we show that $u = u^\star$ satisfies (5.4). Let $v \in K$ be an arbitrary fixed element and consider the function $h$ defined by

$$h(t) = E(u^\star + t(v - u^\star)), \quad t \in [0, 1].$$

Since $K$ is convex, $u^\star + t(v - u^\star) \in K$ for all $t \in [0, 1]$, hence

$$h(t) = E(u^\star + t(v - u^\star)) \geq E(u^\star) = h(0). \tag{5.7}$$

Since

$$h(t) = E(u^\star + t(v - u^\star)) = \langle A(u^\star + t(v - u^\star)), u^\star + t(v - u^\star)\rangle - 2\langle F, u^\star + t(v - u^\star)\rangle =$$
$$t^2\langle A(v - u^\star), v - u^\star\rangle + t[\langle A(v - u^\star), u^\star\rangle + \langle Au^\star, v - u^\star\rangle - 2\langle F, v - u^\star\rangle] +$$
$$\langle Au^\star, u^\star\rangle - \langle F, u^\star\rangle,$$

by (5.7)

$$0 \leq h'(0) = \langle A(v - u^\star), u^\star\rangle + \langle Au^\star, v - u^\star\rangle - 2\langle F, v - u^\star\rangle = 2[\langle Au^\star, v - u^\star\rangle - \langle F, v - u^\star\rangle],$$

so we obtained that $u = u^\star$ satisfies (5.4). Since $A$ is strictly positive, the solution of (5.4) is unique: assuming that $u_j$ satisfies

$$\langle Au_j, v - u_j\rangle \geq \langle F, v - u_j\rangle \text{ for all } v \in K, \quad j = 1, 2,$$

we have

$$\langle Au_1, u_2 - u_1\rangle \geq \langle F, u_2 - u_1\rangle, \quad \langle Au_2, u_1 - u_2\rangle \geq \langle F, u_1 - u_2\rangle.$$

The sum of these inequalities results

$$\langle Au_1 - Au_2, u_2 - u_1\rangle \geq 0, \text{ hence } u_2 = u_1$$

because $A$ is strictly positive. □

As a generalization of (5.4) for arbitrary Banach space $V$ and nonlinear operator $A : V \to V^\star$ we have the definition of an abstract elliptic variational inequality:

**Definition 5.2.** *Let $V$ be a real Banach space, $K \subset V$ a closed convex set, $A : K \to V^\star$ a (nonlinear) operator, $F \in V^\star$. Then the* variational inequality *is the following problem: find $u \in K$ satisfying*

$$\langle A(u), v - u\rangle \geq \langle F, v - u\rangle \text{ for all } v \in K. \tag{5.8}$$

**Remark 5.3.** In general, the variational inequality (5.8) is not connected with the minimum of a functional.

**Remark 5.4.** In the particular case when $K$ is a closed convex cone with the vertex 0, the variational inequality (5.8) holds if and only if

$$\langle Au, v\rangle \geq \langle F, v\rangle \text{ for all } v \in K \text{ and} \tag{5.9}$$

$$\langle Au, u\rangle = \langle F, u\rangle. \tag{5.10}$$

From (5.9) we obtain that in the case $K = V$ (5.8) is equivalent with the equality

$$\langle Au, v\rangle = \langle F, v\rangle \text{ for all } v \in V, \text{ i.e. } A(u) = F.$$

Indeed, from (5.8) with $v = 0$ and $v = 2u$ we obtain

$$-\langle Au, u\rangle \geq -\langle F, u\rangle \text{ and } \langle Au, u\rangle \geq \langle F, u\rangle,$$

respectively, i.e. we have (5.10). Further, subtracting the equality (5.10) from (5.9), we obtain (5.8).

Now we formulate some examples for solutions of (5.8) which can be considered as weak solutions to boundary value problems for equation (1.1) with certain nonlinear boundary conditions.

**Example 5.5.** Consider the linear operator (5.1) defined in $V = H^1(\Omega)$ and set

$$K = \{v \in H^1(\Omega) : v|_{\partial\Omega} \geq 0\}, \quad \langle F, v\rangle = \int_\Omega f v dx \text{ with some } f \in L^2(\Omega).$$

Then $K$ is a closed convex cone with vertex 0.

Now we show that a solution $u \in K$ of (5.8) can be considered as a weak solution of the equation (1.1) with some nonlinear boundary condition. First assume that $u$ is a sufficiently smooth (e.g. $u \in C^2(\overline{\Omega})$) solution of (5.9), (5.10) with sufficiently smooth functions $a_{jk}, c, f$. Then by Gauss's theorem for $v \in K$, $v \in C^1(\overline{\Omega})$

$$\int_\Omega f v dx = \langle F, v\rangle \leq \langle Au, v\rangle = \sum_{j,k=1}^n \int_\Omega a_{jk}(D_k u)(D_j v) dx + \int_\Omega c u v dx = \quad (5.11)$$

$$\int_\Omega v \left[ -\sum_{j,k=1}^n D_j(a_{jk} D_k u) + cu \right] dx + \int_{\partial\Omega} v \sum_{j,k=1}^n a_{jk}(D_k u)\nu_j d\sigma.$$

Setting $v = \varphi$ and $v = -\varphi$ in (5.11) with arbitrary $\varphi \in C_0^1(\Omega)$, we obtain

$$f = -\sum_{j,k=1}^n D_j(a_{jk} D_k u) + cu \text{ in classical sense .} \quad (5.12)$$

Thus (5.11) implies for the "conormal derivative"

$$\partial_\nu^\star u = \sum_{j,k=1}^n a_{jk}(D_k u)\nu_j$$

$$\int_{\partial\Omega} v \partial_\nu^\star u d\sigma = \int_{\partial\Omega} v \sum_{j,k=1}^n a_{jk}(D_k u)\nu_j d\sigma \geq 0$$

for all $v \in C^1(\overline{\Omega})$ with $v|_{\partial\Omega} \geq 0$, hence

$$\partial_\nu^\star u = \sum_{j,k=1}^n a_{jk}(D_k u)\nu_j \geq 0 \text{ on } \partial\Omega \quad (5.13)$$

and by $u \in K$ we have
$$u \geq 0 \text{ on } \partial\Omega. \tag{5.14}$$
Since $\langle Au, u\rangle = \langle F, u\rangle$, we obtain from (5.11)
$$\int_{\partial\Omega} u(\partial_\nu^\star u) d\sigma = 0$$
which implies by (5.13), (5.14)
$$u(\partial_\nu^\star u) = 0 \text{ on } \partial\Omega. \tag{5.15}$$

Summarizing, if $u \in C^2(\overline{\Omega})$ is a solution of the variational inequality (5.8) (i.e. (5.9), (5.10)) then $u$ is a classical solution of the (linear) differential equation (5.12) with the nonlinear boundary conditions (5.13)–(5.15). Conversely, it is easy to show that a solution $u \in C^2(\overline{\Omega})$ of the boundary value problem (5.12)–(5.15) satisfies the variational inequality. Therefore, a function $u \in K$ satisfying the variational inequality (5.8), can be considered as a weak solution of (5.12)–(5.15).

**Example 5.6.** Consider the operator (5.1) in $V = H_0^1(\Omega)$ with
$$K = \{v \in H_0^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}, \quad \langle F, v\rangle = \int_\Omega fv dx \text{ with some } f \in L^2(\Omega).$$
Then $K$ is a closed convex cone with vertex 0.

Assume that $u \in C^2(\overline{\Omega})$ is a solution of (5.8) (i.e. of (5.9) and (5.10)). Let
$$\Omega_+ = \{x \in \Omega : u(x) > 0\}, \quad \Omega_0 = \{x \in \Omega : u(x) = 0\}.$$
Consider an arbitrary function $\varphi \in C_0^1(\Omega_+)$ and let $v = u + \varepsilon\varphi$ with some $\varepsilon \in \mathbb{R}$. Then, clearly, $v \in K$ for sufficiently small $|\varepsilon|$ (because $u$ has a positive minimum on supp$\varphi$) and so from
$$\langle A(u), v - u\rangle \geq \langle F, v - u\rangle$$
we obtain the differential equation (5.10) as in the previous example. Further, since $u \in K$,
$$u = 0 \text{ on } \partial\Omega_+ \tag{5.16}$$
and, clearly,
$$\partial_\nu^\star u = 0 \text{ on } \partial\Omega_+ \cap \partial\Omega_0. \tag{5.17}$$
Thus the smooth solution of (5.8) satisfies
$$-\sum_{j,k=1}^n D_j[a_{jk}D_k u] + cu = f \text{ in } \Omega_+, \quad u > 0 \text{ in } \Omega_+, \tag{5.18}$$
the boundary conditions (5.16), (5.17) and
$$u = 0 \text{ in } \Omega_0 = \Omega \setminus \Omega_+. \tag{5.19}$$
So a smooth solution $u$ ($\in C^2(\overline{\Omega})$) of (5.8) satisfies the boundary value problem (5.16)–(5.19) with “free boundary”.

It is easy to show that if $u \in C^2(\overline{\Omega})$ satisfies (5.16)–(5.19) then $u$ is a solution of (5.8).

## Existence theorems

Now we formulate and prove two existence theorems on the variational inequality (5.8).

**Theorem 5.7.** *Let $V$ be a real reflexive separable Banach space and $K \subset V$ a closed, convex, bounded subset. Assume that $A : K \to V^\star$ is bounded and pseudomonotone. Then for all $F \in V^\star$ there exists $u \in K$ which satisfies (5.8), i.e.*

$$\langle A(u), v - u\rangle \geq \langle F, v - u\rangle \text{ for all } v \in K.$$

**Remark 5.8.** By definition, a bounded operator $A : K \to V^\star$ is called pseudomonotone if

$$(u_k) \to u \text{ weakly in } V, \quad u_k \in K, \quad \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0 \tag{5.20}$$

imply

$$\lim_{k\to\infty} \langle A(u_k), u_k - u\rangle = 0 \text{ and } (A(u_k)) \to A(u) \text{ weakly in } V^\star. \tag{5.21}$$

*Proof of Theorem 5.7.* Let $V_m \subset V$ be linear subspaces of dimension $m$ such that

$$V_1 \subset V_2 \subset ... \subset V_m \subset ... \text{ and } \overline{\cup_{m=1}^\infty V_m} = V.$$

Further, let $K_m = V_m \cap K$. Then $K_m \subset V_m$ is a closed, convex, bounded set,

$$K_1 \subset K_2 \subset ... \subset K_m \subset ... \text{ and } \overline{\cup_{m=1}^\infty K_m} = K.$$

First we show that for all $m$ there exist solutions $u_m \in K_m$ of the ("finite dimensional") variational inequalities

$$\langle A(u_m), v - u_m\rangle \geq \langle F, v - u_m\rangle \text{ for all } v \in K_m. \tag{5.22}$$

In the finite dimensional (Banach) space $V_m$ define some scalar product $[\cdot, \cdot]$ generating a norm which is equivalent with the original norm in $V_m$. If $g \in V^\star$ then the linear functional

$$w \mapsto \langle g, w\rangle, \quad w \in V_m$$

is continuous in the Hilbert space $V_m$ (with the scalar product $[\cdot, \cdot]$), hence there exists a linear and continuous operator $B : V^\star \to V_m$ such that

$$\langle g, w\rangle = [Bg, w] \text{ for all } w \in V_m.$$

Thus the inequality (5.22) can be written in the form

$$[B(A(u_m)), v - u_n] \geq [BF, v - u_m], \quad v \in K_m,$$

i.e.

$$[u_m, v - u_m] \geq [u_m + BF - B(A(u_m)), v - u_m], \quad v \in K_m. \tag{5.23}$$

Denote by $P_m$ the operator, projecting $V_m$ on to the convex set $K_m$ with respect to the scalar product $[\cdot,\cdot]$. Then inequality (5.23) is equivalent with

$$u_m = P_m(u_m + BF - B[A(u_m)]). \tag{5.24}$$

Consider the operator $Q_m : K_m \to K_m$, defined by

$$Q_m(v) = P_m(v + BF - B[A(v)]), \quad v \in K_m. \tag{5.25}$$

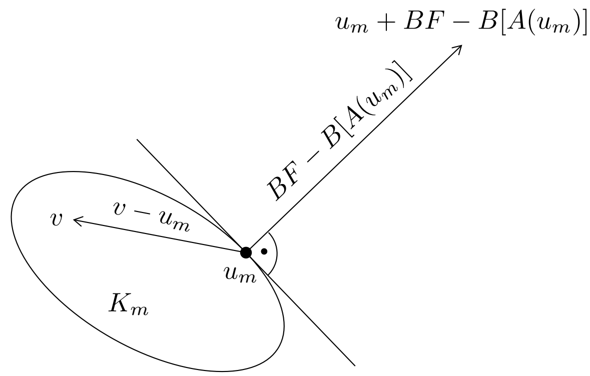


Figure 1.1: Inequality (5.23)

We claim that $Q_m$ is continuous. It is sufficient to show weak continuity, as $K_n$ is of finite dimension. Assume that $(v_k) \to v$ in $K_m$. Since the bounded operator $A$ is pseudomonotone, $A$ is demicontinuous (Proposition 2.7), thus

$$(A(v_k)) \to A(v) \text{ weakly in } V^\star \text{ and so } B[A(v_k)] \to B[A(v)] \text{ in } K_m \text{ and}$$

$$P_m(v_k + BF - B[A(v_k)]) \to P_m(v + BF - B[A(v)]) \text{ as } k \to \infty.$$

Brouwer's fixed point theorem implies that the continuous map $Q_m : K_m \to K_m$ has a fixed point, i.e. there is a solution $u_m$ of (5.24).

Now consider the sequence $(u_m)$ of solutions to (5.24) (i.e. to (5.22)). Since $u_m \in K_m \subset K$, $K$ is bounded and $V$ is reflexive, there is a subsequence of $(u_m)$, again denoted by $(u_m)$ such that

$$(u_m) \to u \text{ weakly in } V. \tag{5.26}$$

Since $u_m \in K$, $K$ is convex and closed, we have $u \in K$. Now we prove

$$\limsup_{m\to\infty} \langle A(u_m), u_m - u \rangle \leq 0. \tag{5.27}$$

As $\cup_{m=1}^{\infty} K_m$ is dense in $K$, for arbitrary $\varepsilon > 0$ there is $u_0 \in \cup_{m=1}^{\infty} K_m$ such that

$$\|u - u_0\|_V \leq \varepsilon. \tag{5.28}$$

Further, $u_0 \in K_m$ for sufficiently large $m$, thus by (5.22)

$$\langle A(u_m), u_m - u_0 \rangle \leq \langle F, u_m - u_0 \rangle,$$

hence by (5.28) and the boundedness of $\|A(u_m)\|_{V^\star}$

$$\langle A(u_m), u_m - u \rangle = \langle A(u_m), u_m - u_0 \rangle + \langle A(u_m), u_0 - u \rangle \leq \langle F, u_m - u_0 \rangle + c\varepsilon$$

with some constant $c$. By (5.26), (5.28), this inequality implies (5.27).

Finally, since $A$ is pseudomonotone, (5.26), (5.27) imply

$$\lim_{m \to \infty} \langle A(u_m), u_m - u \rangle = 0, \quad (A(u_m)) \to A(u) \text{ weakly in } V^\star \tag{5.29}$$

(for a subsequence). For arbitrary fixed $v \in \cup_{m=1}^{\infty} K_m$ the variational inequalities (5.22) can be written in the form

$$\langle A(u_m), v - u \rangle + \langle A(u_m), u - u_m \rangle \geq \langle F, v - u_m \rangle \text{ if } m \text{ is sufficiently large.}$$

By (5.26), (5.29), from this inequality we obtain as $n \to \infty$

$$\langle A(u), v - u \rangle \geq \langle F, v - u \rangle \text{ for any } v \in \cup_{m=1}^{\infty} K_m. \tag{5.30}$$

Since $\cup_{m=1}^{\infty} K_m$ is dense in $K$, (5.30) holds for arbitrary $v \in K$, i.e. $u$ is a solution of (5.8). □

Now we formulate the extension of Theorem 5.7 to unbounded sets $K$.

**Theorem 5.9.** *Let $V$ be a reflexive separable Banach space and $K \subset V$ a closed, convex subset. Assume that $A : K \to V^\star$ is bounded, pseudomonotone and coercive in the following sense: there exists $v_0 \in K$ such that*

$$\frac{\langle A(v), v - v_0 \rangle}{\|v\|_V} \to +\infty \text{ if } \|v\|_V \to \infty, \quad v \in K. \tag{5.31}$$

*Then for arbitrary $F \in V^\star$ there exists a solution $u \in K$ of (5.8).*

*Proof.* Set $B_R = \{v \in V : \|v\| \leq R\}$ and $K_R = K \cap B_R$. Since $K_R$ is a closed, convex, bounded set, by Theorem 5.7 there exists $u_R \in K_R$ with

$$\langle A(u_R), v - u_R \rangle \geq \langle F, v - u_R \rangle \text{ for any } v \in K_R. \tag{5.32}$$

Applying (5.32) to $v = v_0$ and $R \geq \|v_0\|_V$, we obtain by (5.31)

$$\langle A(u_R), v_0 - u_R \rangle \geq \langle F, v_0 - u_R \rangle \geq -\|F\|_{V^\star} \|v_0 - u_R\|_V,$$

hence

$$\frac{\langle A(u_R), u_R - v_0 \rangle}{\|u_R\|_V} \leq \|F\|_{V^\star} \frac{\|v_0 - u_R\|_V}{\|u_R\|_V} \leq \|F\|_{V^\star} \frac{\|v_0\|_V + \|u_R\|_V}{\|u_R\|_V}$$

where the right hand side is bounded if $\|u_R\|_V \geq 1$. Thus by (5.31) $\|u_R\|_V$ is bounded for all $R$. Consequently, there are a sequence $(R_k)$, converging to $+\infty$ and $u \in V$ such that

$$(u_{R_k}) \to u \text{ weakly in } V. \tag{5.33}$$

Since $u_{R_k} \in K_{R_k} \subset K$, we have $u \in K$. According to (5.32), for any $v \in K$, sufficiently large $k$

$$\langle A(u_{R_k}), u_{R_k} - u\rangle \leq \langle F, u_{R_k} - u\rangle \to 0$$

thus

$$\limsup_{k\to\infty} \langle A(u_{R_k}), u_{R_k} - u\rangle \leq 0,$$

hence by (5.33)

$$\lim_{k\to\infty} \langle A(u_{R_k}), u_{R_k} - u\rangle = 0 \text{ and } (A(u_{R_k})) \to A(u) \text{ weakly in } V^\star \tag{5.34}$$

because $A$ is pseudomonotone.

Applying (5.32) with arbitrary fixed $v \in K$, $R = R_k > \|v\|_V$, we obtain

$$\langle A(u_{R_k}), v - u\rangle + \langle A(u_{R_k}), u - u_{R_k}\rangle \geq \langle F, v - u_{R_k}\rangle$$

whence one obtains (by (5.33), (5.34)) as $k \to \infty$

$$\langle A(u), v - u\rangle \geq \langle F, v - u\rangle,$$

i.e. $u \in K$ satisfies (5.8). $\square$

**Remark 5.10.** If $A : K \to V^\star$ is strictly monotone then the solution of (5.8) is unique.

Indeed, assuming that $u_j \in K$ satisfies

$$\langle A(u_j), v - u_j\rangle \geq \langle F, v - u_j\rangle, \text{ for all } v \in K, \quad j = 1, 2,$$

we obtain

$$\langle A(u_1), u_2 - u_1\rangle \geq \langle F, u_2 - u_1\rangle, \quad \langle A(u_2), u_1 - u_2\rangle \geq \langle F, u_1 - u_2\rangle$$

whence

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle \leq 0$$

which implies $u_1 = u_2$.

**Remark 5.11.** Similarly to Remark 2.17, it is easy to show that if $A$ is uniformly monotone then the solution $u$ of (5.8) depends on $F$ continuously. Indeed, assuming

$$\langle A(u_j), v - u_j\rangle \geq \langle F_j, v - u_j\rangle, \text{ for all } v \in K, \quad j = 1, 2,$$

we have

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \leq \langle F_1 - F_2, u_1 - u_2 \rangle \leq \|F_1 - F_2\|_{V^\star} \|u_1 - u_2\|_V.$$

If $A$ is uniformly monotone then according to Definition 2.15

$$a(\|u_1 - u_2\|_V)\|u_1 - u_2\|_V \leq \langle A(u_1) - A(u_2), u_1 - u_2 \rangle,$$

thus

$$a(\|u_1 - u_2\|_V) \leq \|F_1 - F_2\|_{V^\star}, \text{ i.e. } \|u_1 - u_2\|_V \leq a^{-1}(\|F_1 - F_2\|_{V^\star})$$

where $a^{-1} : [0, \infty) \to [0, \infty)$ is a continuous function and $a^{-1}(0) = 0$.

## Problems

1. Consider the operator (5.1) in $V = H_0^1(\Omega)$ with

$$K = \{v \in H_0^1(\Omega) : \psi_1 \leq v \leq \psi_2 \text{ a.e. in } \Omega\}$$

where $\psi_1, \psi_2$ are measurable functions. By using the arguments in Example 5.6, show that in this case the solution of the variational inequality (5.8) can be considered as a weak solution of certain boundary value problem with "free boundary".

2. Consider the operator (5.1) in $V = H_0^1(\Omega)$ with

$$K = \{v \in H_0^1(\Omega) : |Dv(x)| \leq 1 \text{ a.e. in } \Omega\}$$

By using the arguments in Example 5.6, show that in this case the solution of the variational inequality (5.8) can be considered as a weak solution of certain boundary value problem with "free boundary".

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $V = W_0^{1,p}(\Omega)$, $p \geq 2$ and $K \subset V$ a closed convex set. Define the operator $A$ by

$$\langle A(u), v \rangle = \sum_{j=1}^{n} \int_\Omega (D_j u)|Du|^{p-2} D_j v, \quad u, v \in V.$$

Prove that then for all $F \in V^\star$ there exists a unique solution of the variational inequality (5.8) and it depends on $F$ continuously.

4. Let $V$ be a closed linear subspace of $W^{m,p}(\Omega)$ $(m \geq 1, p \geq 2)$ and $K \subset V$ a closed convex set. Define the operator $A$ by

$$\langle A(u), v \rangle = \sum_{|\alpha| \leq m} \int_\Omega (D^\alpha u)|D^\alpha u|^{p-2} D^\alpha v, \quad u, v \in V.$$

Show that for all $F \in V^\star$ there exists a unique solution of the variational inequality (5.8) and it depends on $F$ continuously.

# Chapter 2

# FIRST ORDER EVOLUTION EQUATIONS

## 6 Formulation of the abstract problem

In this section we shall motivate and formulate the abstract Cauchy problem for first order evolution equations and problems which will be considered for nonlinear parabolic equations with nonlinear elliptic operators of "divergence type".

In [67] the linear parabolic equation of the following form was considered:

$$D_t u - \sum_{j,k=1}^{n} D_j[a_{jk} D_k u] + cu = f \text{ in } Q_T = (0,T) \times \Omega \tag{6.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $D_t = \frac{\partial}{\partial t}$, with the Dirichlet boundary condition

$$u|_{\Gamma_T} = g \text{ where } \Gamma_T = [0,T) \times \partial\Omega \tag{6.2}$$

and the initial condition

$$u(0,x) = h(x), \quad x \in \Omega. \tag{6.3}$$

Assume that $u \in C^{1,2}(\overline{Q_T})$ (i.e. $u$ is a function which is once continuously differentiable with respect to $t$ and twice continuously differentiable with respect to $x$ in $\overline{Q_T}$) is a classical solution of (6.1) – (6.3). Multiplying the differential equation (6.1) with a test function $v \in C^1(\overline{Q_T})$ and integrating over $Q_T$, by Gauss theorem we obtained an equation which (with (6.2)) defined the weak solution of problem (6.1) – (6.3). In this formulation the equation contained the initial condition (6.3), too.

Now we shall give another definition of the weak solution for certain nonlinear parabolic equations and as a particular case for the linear equation (6.1). We

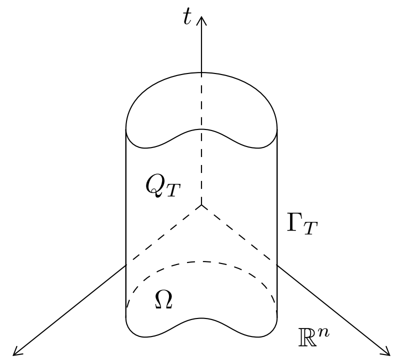


Figure 2.1: The “cylinder” $Q_T$

shall consider nonlinear parabolic equations of the form

$$D_t u - \sum_{j=1}^{n} D_j[a_j(t,x,u,Du)] + a_0(t,x,u,Du) = f \text{ in } Q_T, \tag{6.4}$$

which is analogous to the nonlinear elliptic equation (1.4) of divergence form.

In order to define the weak solution of (6.4), (6.2), (6.3) with homogeneous boundary condition, multiply the differential equation (6.4) with a test function $v \in C_0^1(\Omega)$ (i.e. by a $C^1$ function with compact support), to obtain

$$\int_\Omega (D_t u) v dx + \sum_{j=1}^{n} \int_\Omega a_j(t,x,u,Du) D_j v dx + \int_\Omega a_0(t,x,u,Du) v dx = \tag{6.5}$$

$$\int_\Omega f v dx.$$

Later we shall see that if the functions $a_j$ satisfy certain growth conditions (which are analogous to $(A2)$) then for a.a. fixed $t \in [0,T]$,

$$x \mapsto a_j(t,x,u(t,x),Du(t,x)) \in L^q(\Omega) \text{ if } x \mapsto u(t,x) \in W^{1,p}(\Omega)$$

$$(1 < p < \infty, 1/p + 1/q = 1).$$

Then (6.5) holds for all test functions $v \in W_0^{1,p}(\Omega)$.

Introduce the notations

$$V = W_0^{1,p}(\Omega), \quad U(t) = x \mapsto u(t,x), \quad x \in \Omega$$

and with a fixed $t \in [0,T]$ define operator $\tilde{A}(t)$ and operator $A$ by

$$\langle [A(U)](t), v \rangle = \langle [\tilde{A}(t)][U(t)], v \rangle = \tag{6.6}$$

$$\sum_{j=1}^{n} \int_\Omega a_j(t,x,u,Du) D_j v dx + \int_\Omega a_0(t,x,u,Du) v dx, \quad U(t), v \in V;$$

and define $F(t)$ for all fixed $t \in [0,T]$ by

$$F(t)v = \int_\Omega f(t,x)v(x)dx, \text{ assuming } x \mapsto f(t,x) \in L^q(\Omega). \tag{6.7}$$

Then for each fixed $t \in [0,T]$

$$[A(U)](t) \in V^\star, \quad \tilde{A}(t) : V \to V^\star, \quad F(t) \in V^\star$$

and equation (6.5) can be written in the form of the "ordinary differential equation"

$$[D_t U](t) + [A(U)](t) = F(t), \quad t \in [0,T]. \tag{6.8}$$

In order to give the exact definition of the equation (6.8), we have to define the derivative $D_t U$. Further, we have to give the exact definition of the initial condition $U(0) = h$, corresponding to (6.3). The homogeneous boundary condition (6.2) (i.e. the case $g = 0$) will be taken into consideration by $V = W_0^{1,p}(\Omega)$.

First we define the function spaces $L^p(0,T;V)$ which will be the domain of definition of operator $A$.

**Definition 6.1.** *Let $V$ be a Banach space, $0 < T < \infty$, $1 \le p < \infty$. Denote by $L^p(0,T;V)$ the set of measurable functions $f : (0,T) \to V$ such that $\|f(t)\|_V^p$ is integrable and define the norm by*

$$\|f\|^p_{L^p(0,T;V)} = \int_0^T \|f(t)\|_V^p dt.$$

Then $L^p(0,T;V)$ is a Banach space over $\mathbb{R}$ (identifying functions that are equal almost everywhere on $(0,T)$). If $V$ is separable then $L^p(0,T;V)$ is separable, too.

Denoting by $V^\star$ the dual space of $V$ and by $\langle \cdot,\cdot \rangle$ the dualities in spaces $V^\star$, $V$, we have for all $f \in L^p(0,T;V)$, $g \in L^q(0,T;V^\star)$ with $1 < p < \infty$, $1/p + 1/q = 1$ Hölder's inequality

$$\left| \int_0^T \langle g(t), f(t) \rangle dt \right| \le \left[ \int_0^T \|g(t)\|^q_{V^\star} dt \right]^{1/q} \left[ \int_0^T \|f(t)\|^p_V dt \right]^{1/p}.$$

Further, for $1 < p < \infty$ the dual space of $L^p(0,T;V)$ is isomorphic and isometric to $L^q(0,T;V^\star)$. Thus we may identify the dual space of $L^p(0,T;V)$ with $L^q(0,T;V^\star)$. Consequently, if $V$ is reflexive then $L^p(0,T;V)$ is reflexive for $1 < p < \infty$. The detailed proof of the above facts can be found, e.g., in [93]. The dualities between $L^q(0,T;V^\star)$ and $L^p(0,T;V)$ will be denoted by $[\cdot,\cdot]$.

**Definition 6.2.** *Let $V$ be a real separable and reflexive Banach space and $H$ a real separable Hilbert space with the scalar product $(\cdot,\cdot)$ such that the imbedding $V \subset H$ is continuous and $V$ is dense in $H$. Then the formula*

$$\langle \tilde{v}, u \rangle = (v,u), \quad u \in V, \quad v \in H$$

*defines a linear continuous functional $\tilde{v}$ over $V$ and it generates a bijection between $H$ and a subset of $V^\star$, i.e. we may write*

$$V \subset H \subset V^\star$$

*which will be called an* evolution triple.

**Example 6.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $m$ a nonnegative integer and $2 \le p < \infty$. Let $V$ be a closed linear subspace of the Sobolev space $W^{m,p}(\Omega)$ and $H = L^2(\Omega)$. Then $V \subset H \subset V^\star$ is an evolution triple.

Now we define the generalized derivatives of functions $u \in L^p(0,T;V)$.

**Definition 6.4.** *Let $V \subset H \subset V^\star$ be an evolution triple, $u \in L^p(0,T;V)$. If there exists $w \in L^q(0,T;V^\star)$ such that*

$$\int_0^T \varphi'(t)u(t)dt = -\int_0^T \varphi(t)w(t)dt$$

*for all $\varphi \in C_0^\infty(0,T)$ (i.e. for all infinitely many times differentiable functions on $(0,T)$ with compact support) then $w$ is called the* generalized derivative *of $u$ and it is denoted by $u'$.*

**Remark 6.5.** In the above equality $u(t) \in V$ is considered as an element of $V^\star$. In this case we shall write briefly $u' \in L^q(0,T;V^\star)$. It is easily seen that the generalized derivative is unique.

Further, it is not difficult to show that $u' = w \in L^q(0,T;V^\star)$ if and only if

$$\int_0^T (u(t),v)_H \varphi'(t)dt = -\int_0^T \langle w(t),v\rangle \varphi(t)dt \text{ for all } \varphi \in C_0^\infty(0,T), \quad v \in H.$$

**Theorem 6.6.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $1/p+1/q = 1$, $0 < T < \infty$. Then*

$$W_p^1(0,T;V,H) = \{u \in L^p(0,T;V) : u' \in L^q(0,T;V^\star)\}$$

*with the norm*

$$\|u\| = \|u\|_{L^p(0,T;V)} + \|u'\|_{L^q(0,T;V^\star)}$$

*is a Banach space. $W_p^1(0,T;V,H)$ is continuously imbedded into $C([0,T];H)$ (the space of continuous functions $v : [0,T] \to H$ with the supremum norm) in the following sense: to every $u \in W_p^1(0,T;V,H)$ there is a uniquely defined $\tilde{u} \in C([0,T];H)$ such that $u(t) = \tilde{u}(t)$ for a.e. $t \in [0,T]$ and*

$$\|\tilde{u}\|_{C([0,T];H)} \le const\|u\|_{W_p^1(0,T;V,H)}.$$

*Further, the following integration by parts formula holds for arbitrary functions $u, v \in W_p^1(0,T;V,H)$ and $0 \le s < t \le T$:*

$$(u(t),v(t)) - (u(s),v(s)) = \int_s^t [\langle u'(\tau),v(\tau)\rangle + \langle v'(\tau),u(\tau)\rangle]d\tau. \tag{6.9}$$

(In (6.9) $u(t), u(s)$ mean the values of the above $\tilde{u} \in C([0,T];H)$ in $t, s$, respectively.)

**Remark 6.7.** In the case $v = u \in W_p^1(0,T;V,H)$ we obtain from (6.9)

$$\|u(t)\|_H^2 - \|u(s)\|_H^2 = 2\int_s^t \langle u'(\tau), u(\tau)\rangle d\tau.$$

The detailed proof of Theorem 6.6 can be found in [30].

# 7 Cauchy problem with monotone operators

In this section let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$ and let us use the notations

$$X = L^p(0,T;V), \quad [F,v] = \int_0^T \langle F(t), v(t)\rangle dt, \quad v \in X, F \in X^\star.$$

Let $A : X \to X^\star$ be an operator given by

$$[A(u)](t) = [\tilde{A}(t)](u(t))$$

where for a.a. fixed $t \in [0,T]$, $\tilde{A}(t)$ maps $V$ into $V^\star$, $u_0 \in H$, $F \in X^\star$. We want to find $u \in W_p^1(0,T;V,H)$ satisfying

$$u' + A(u) = F, \quad u(0) = u_0. \tag{7.1}$$

By Theorem 6.6 the initial condition makes sense.

**Theorem 7.1.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$. Assume that for all fixed $t \in [0,T]$, $\tilde{A}(t) : V \to V^\star$ is monotone, hemicontinuous and bounded in the sense*

$$\|\tilde{A}(t)(v)\|_{V^\star} \le c_1\|v\|_V^{p-1} + k_1(t) \tag{7.2}$$

*for all $v \in V$, $t \in [0,T]$ with a suitable constant $c_1$ and a function $k_1 \in L^q(0,T)$. Further, $\tilde{A}(t)$ is coercive in the sense: there exist a constant $c_2 > 0$ and a function $k_2 \in L^1(0,T)$ such that*

$$\langle \tilde{A}(t)(v), v\rangle \ge c_2\|v\|_V^p - k_2(t) \tag{7.3}$$

*for all $v \in V$, $t \in [0,T]$. Finally, for arbitrary fixed $u, v \in V$, the function*

$$t \mapsto \langle \tilde{A}(t)(u), v\rangle, \quad t \in [0,T] \text{ is measurable }. \tag{7.4}$$

*Then for arbitrary $F \in L^q(0,T;V^\star)$ and $u_0 \in H$ there exists a unique solution of problem (7.1) with the operator $A$ defined by $[A(u)](t) = [\tilde{A}(t)](u(t))$.*

In the proof we shall apply the following theorem of Carathéodory (see [93] and [19]).

**Theorem 7.2.** *Set $I = [t_0, t_0 + r]$, $K = \{x \in \mathbb{R}^n : |x - x_0| \leq r\}$ and assume that the functions $f_j : I \times K \to \mathbb{R}$, $j = 1, \dots, n$ satisfy the following conditions:*

$$t \mapsto f_j(t, x) \text{ is measurable on } I \text{ for all fixed } x \in K,$$

$$x \mapsto f_j(t, x) \text{ is continuous on } K \text{ for a.a. } t \in I$$

*("Carathéodory conditions") and there exists a function $M \in L^1(I)$ such that*

$$|f_j(t, x)| \leq M(t) \text{ for all } x \in K, \text{ a.a. } t \in I.$$

*Then there exist absolute continuous functions $\xi_j$ satisfying the initial value problem*

$$\xi_j'(t) = f_j(t, \xi(t)) \text{ a.e. in a neighbourhood of } t_0, \quad \xi(0) = x_0$$

*where $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$.*

*Proof of Theorem 7.1.* The proof is based on Galerkin's approximation. Since $V$ is separable, there exists a countable set of linearly independent elements $z_1, ..., z_k, ...$ such that their finite linear combinations are dense in $V$. We shall find the $m$-th approximation of a solution $u$ in the form

$$u_m(t) = \sum_{k=1}^{m} a_{km}(t) z_k \text{ with some } a_{km} \in W^{1,q}(0, T)$$

such that for a.e. $t \in [0, T]$

$$\langle u_m'(t), z_j \rangle + \langle \tilde{A}(t)[u_m(t)], z_j \rangle = \langle F(t), z_j \rangle, \quad j = 1, \dots, m, \tag{7.5}$$

$$u_m(0) = u_{m0} \in V_n = \text{span}(z_1, ..., z_m), \text{ where } (u_{m0}) \to u_0 \text{ in } H. \tag{7.6}$$

System (7.5) is a system of ordinary differential equations for $a_{km}$ because it has the form

$$\sum_{k=1}^{m} a_{km}'(t)(z_k, z_j) + \langle \tilde{A}(t)[\sum_{k=1}^{m} a_{km}(t) z_k], z_j \rangle = \langle F(t), z_j \rangle \tag{7.7}$$

and (7.6) is equivalent to

$$a_{jm}(0) = \alpha_{j0}, \quad j = 1, \dots, m \tag{7.8}$$

with some $\alpha_{j0} \in \mathbb{R}$. The system (7.7) can be transformed to explicit form since the determinant $\det(z_k, z_j) \neq 0$, because $z_1, ..., z_m$ are linearly independent.

According to assumption (7.4), the functions

$$a_j(t, w) = a_j(t, w_1, ..., w_m) = \langle \tilde{A}(t) \Big[ \sum_{k=1}^{m} w_k z_k \Big], z_j \rangle \quad j = 1, \dots, m$$

are measurable in $t$ (with fixed $w$) and continuous in $w = (w_1, ..., w_m)$, because for all fixed $t \in [0, T]$, $\tilde{A}(t) : V \to V^\star$ is monotone, hemicontinuous, bounded by the assumptions of the theorem, thus it is pseudomonotone and so it is demicontinuous (see Propositions 2.5, 2.7). From (7.2) it follows that $|a_j(t, w)|$ can be estimated locally by an integrable function $M(t)$. Consequently, by Theorem 7.2 (theorem of Carathéodory ), there exists a solution of (7.7) in a neighbourhood of 0.

The coercivity assumption (7.3) implies that the solutions $a_{jn}$ and thus $u_n$ can be extended to the whole interval $[0, T]$. Indeed, if $u_m$ satisfies (7.5) in a neighbourhood of 0, then multiplying (7.5) by $a_{jm}(t)$ and summing with respect to $j$, we obtain

$$\langle u_m'(t), u_m(t)\rangle + \langle [\tilde{A}(t)][u_m(t)], u_m(t)\rangle = \langle F(t), u_m(t)\rangle. \tag{7.9}$$

Integrating (7.9) over an interval $(0, t)$ $(t \in [0, T])$, by Remark 6.7 one obtains

$$\frac{1}{2}\|u_m(t)\|_H^2 - \frac{1}{2}\|u_m(0)\|_H^2 + \int_0^t \langle [\tilde{A}(\tau)][u_m(\tau)], u_m(\tau)\rangle d\tau = \tag{7.10}$$

$$\int_0^t \langle F(\tau), u_m(\tau)\rangle d\tau,$$

hence by (7.3)

$$\frac{1}{2}\|u_m(t)\|_H^2 + c_2 \int_0^t \|u_m(\tau)\|_V^p d\tau \le \frac{1}{2}\|u_{m0}\|_H^2 + \tag{7.11}$$

$$\int_0^T k_2(\tau)d\tau + \|F\|_{L^q(0,T;V^\star)} \left\{\int_0^t \|u_m(\tau)\|_V^p d\tau\right\}^{1/p}.$$

As the constant $c_2$ is positive and $p > 1$, we get from (7.11) that there is a constant with

$$\int_0^t \|u_m(\tau)\|_V^p d\tau \le \text{const}, \quad t \in [0, T] \tag{7.12}$$

and thus

$$\|u_m(t)\|_H^2 \le \text{const}, \quad t \in [0, T]. \tag{7.13}$$

Consequently, $a_{jm}(t)$ (defined in a neighbourhood of 0) can be estimated by a constant, not depending on $t$, therefore, the solutions $a_{jm}$ can be extended to $[0, T]$.

Further, by using the notations $X = L^p(0, T; V)$, $X^\star = L^q(0, T; V^\star)$, we obtain that

$$\|u_m\|_X, \quad \sup_{t\in[0,T]} \|u_m(t)\|_H, \quad m = 1, 2, ... \text{ are bounded,} \tag{7.14}$$

hence $\|A(u_m)\|_{X^\star}$ is bounded, too because by (7.2) $A : X \to X^\star$ is a bounded operator. Since $X, X^\star$ and $H$ are reflexive, there exist a subsequence of $(u_m)$, again denoted by $(u_m)$, and $u \in X$, $w \in X^\star$, $z \in H$ such that

$$(u_m) \to u \text{ weakly in } X, \quad (A(u_m)) \to w \text{ weakly in } X^\star, \tag{7.15}$$

$$(u_m(T)) \to z \text{ weakly in } H.$$

Now we prove

**Lemma 7.3.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$. Assume that $u_m$ satisfies (7.5), $(u_m) \to u$ weakly in $L^p(0,T;V)$, $(A(u_m)) \to w$ weakly in $X^\star$, $(u_m(0)) \to u_0$ weakly in $H$ and $(u_m(T)) \to z$ weakly in $H$. Then*

$$u' \in L^q(0,T;V^\star), \quad u' + w = F, \quad u(0) = u_0, \quad u(T) = z. \tag{7.16}$$

*Proof.* Let $\psi \in C^\infty[0,T]$ be an arbitrary function and $v \in V$ an arbitrary element. Since $\overline{\cup_{l=1}^\infty V_l} = V$, there exist

$$v_l \in V_l \text{ such that } (v_l) \to v \text{ in } V. \tag{7.17}$$

Clearly, $\psi v_l \in W^1_p(0,T;V,H)$, $u_m \in W^1_p(0,T;V,H)$, thus by (6.9), (7.5)

$$(u_m(T), \psi(T)v_l) - (u_m(0), \psi(0)v_l) = \tag{7.18}$$

$$\int_0^T [\langle u_m'(t), \psi(t)v_l\rangle + \langle \psi'(t)v_l, u_m(t)\rangle]dt =$$

$$\int_0^T [\langle F(t) - [\tilde{A}(t)][u_m(t)], \psi(t)v_l\rangle + \langle \psi'(t)v_l, u_m(t)\rangle]dt.$$

By the assumption of the lemma we obtain from (7.18) as $m \to \infty$

$$(z, \psi(T)v_l) - (u_0, \psi(0)v_l) =$$

$$\int_0^T [\langle F(t) - w, \psi(t)v_l\rangle + \langle \psi'(t)v_l, u(t)\rangle]dt.$$

Thus by (7.17) we get as $l \to \infty$

$$(z, \psi(T)v) - (u_0, \psi(0)v) = \tag{7.19}$$

$$\int_0^T [\langle F(t) - w, \psi(t)v\rangle + \langle \psi'(t)v, u(t)\rangle]dt.$$

In the case $\psi \in C_0^\infty(0,T)$ (7.19) implies

$$\int_0^T [\langle F(t) - w, v\rangle \psi(t)dt = -\int_0^T \langle v, u(t)\rangle \psi'(t)dt$$

thus by Remark 6.5 there exists $u' \in L^q(0,T;V^\star)$ and

$$u'(t) = F(t) - w, \quad u \in W^1_p(0,T;V,H). \tag{7.20}$$

Due to (6.9), (7.19), (7.20) for all $v$

$$(u(T), \psi(T)v) - (u(0), \psi(0)v) = \int_0^T [\langle u'(t), \psi(t)v\rangle + \langle \psi'(t)v, u(t)\rangle]dt = \tag{7.21}$$

$$(z, \psi(T)v) - (u_0, \psi(0)v).$$

Hence with a function $\psi \in C^\infty[0,T]$, $\psi(T) = 1$, $\psi(0) = 0$ we obtain $u(T) = z$, and with $\psi(T) = 0$, $\psi(0) = 1$, $u(0) = u_0$. So by (7.20) we have proved Lemma 7.3. □

By (7.6) and Lemma 7.3 (7.5) implies (7.16). Further, we show

$$\limsup_{m\to\infty} [A(u_m), u_m - u] \le 0. \tag{7.22}$$

By (7.10)

$$\int_0^T \langle [\tilde{A}(t)][u_m(t)], u_m(t)\rangle dt = \int_0^T \langle F(t), u_m(t)\rangle dt + \frac{1}{2}\|u_m(0)\|_H^2 - \frac{1}{2}\|u_m(T)\|_H^2,$$

hence (7.6), (7.15), (7.16) imply

$$\limsup_{m\to\infty} \int_0^T \langle [\tilde{A}(t)][u_m(t)], u_m(t)\rangle dt = \tag{7.23}$$

$$\int_0^T \langle F(t), u(t)\rangle dt + \frac{1}{2}\|u(0)\|_H^2 - \frac{1}{2}\liminf_{m\to\infty} \|u_m(T)\|_H^2.$$

Since by (7.16) in the Hilbert space $H$

$$u_m(T) \to u(T) \text{ weakly in } H,$$

we have

$$\|u(T)\|_H \le \liminf_{m\to\infty} \|u_m(T)\|_H,$$

whence (7.16), (7.23), Remark 6.7 imply

$$\limsup_{m\to\infty} [A(u_m), u_m] \le [F, u] + \frac{1}{2}\|u(0)\|_H^2 - \frac{1}{2}\|u(T)\|_H^2 =$$

$$[u', u] + [w, u] + \frac{1}{2}\|u(0)\|_H^2 - \frac{1}{2}\|u(T)\|_H^2 = [w, u],$$

thus by (7.15)

$$\limsup_{m\to\infty} [A(u_m), u_m - u] \le [w, u] - [w, u] = 0,$$

i.e. we have (7.22).

Finally, by (7.2) $A : X \to X^\star$ is bounded and it is monotone since $\tilde{A}(t)$ is monotone for each fixed $t$. Because of the hemicontinuity of $\tilde{A}(t)$, $A : X \to X^\star$ is hemicontinuous by (7.2) and Lebesgue's dominated convergence theorem. Therefore, Proposition 2.5 implies that $A : X \to X^\star$ is pseudomonotone ($X = L^p(0,T;V)$ is reflexive). Consequently, (7.15), (7.22) imply $w = A(u)$ which completes the proof of the existence.

Uniqueness of the solution follows from the fact that $\tilde{A}(t) : V \to V^\star$ is monotone for all $t$. Indeed, assuming that $u_1, u_2 \in W_p^1(0,T;V,H)$ are solutions of (7.1), we find for all $t \in [0,T]$

$$\int_0^t \langle u_i'(\tau), u_1(\tau) - u_2(\tau)\rangle d\tau + \int_0^t \langle [A(u_i)](\tau), u_1(\tau) - u_2(\tau)\rangle d\tau =$$

$$\int_0^t \langle F(\tau), u_1(\tau) - u_2(\tau)\rangle d\tau, \quad i = 1,2$$

whence

$$\int_0^t \langle u_1'(\tau) - u_2'(\tau), u_1(\tau) - u_2(\tau)\rangle d\tau + \tag{7.24}$$

$$\int_0^t \langle [A(u_1)](\tau) - [A(u_2)](\tau), u_1(\tau) - u_2(\tau)\rangle d\tau = 0.$$

Since $\tilde{A}(\tau)$ is monotone for a.a. fixed $\tau$, the second term on the left hand side of (7.24) is nonnegative, thus by (6.9)

$$\|u_1(t) - u_2(t)\|_H^2 - \|u_1(0) - u_2(0)\|_H^2 \le 0$$

which implies $\|u_1(t) - u_2(t)\|_H \le 0$ for each $t$ because $u_1(0) - u_2(0) = 0$, thus $u_1 = u_2$. □

**Remark 7.4.** Assume that the conditions of Theorem 7.1 are satisfied such that $\tilde{A}(t)$ is uniformly monotone in the sense

$$\langle [\tilde{A}(t)](v_1) - [\tilde{A}(t)](v_2), v_1 - v_2\rangle \ge c\|v_1 - v_2\|_V^p, \quad v_1, v_2 \in V \tag{7.25}$$

with some constant $c > 0$, for all $t \in [0,T]$. Then the solution of (7.1) depends on $F$ and $u_0$ continuously: if $u_j$ is a solution of (7.1) with $F = F_j$, $u_0 = u_{0j}$ $(j = 1,2)$ then for all $t \in [0,T]$

$$\|u_1(t) - u_2(t)\|_H^2 + c\|u_1 - u_2\|_{L^p(0,T;V)}^p \le \tag{7.26}$$

$$\tilde{c}\|F_1 - F_2\|_{L^q(0,T;V^\star)}^q + \|u_{01} - u_{02}\|_H^2$$

with some positive constant $\tilde{c}$. Indeed, similarly to (7.24) we obtain

$$\|u_1(t) - u_2(t)\|_H^2 - \|u_{01} - u_{02}\|_H^2 + 2c\int_0^t \|u_1(\tau) - u_2(\tau)\|_V^p d\tau \le$$

$$2\left\{\int_0^t \|F_1(\tau) - F_2(\tau)\|_{V^\star}^q d\tau\right\}^{1/q} \left\{\int_0^t \|u_1(\tau) - u_2(\tau)\|_V^p d\tau\right\}^{1/p}$$

whence, by using Young's inequality with a sufficiently small $\varepsilon > 0$ we obtain (7.26).

**Remark 7.5.** Assume that there exists $a > 0$ such that the operator $B$, defined by $B(v) = [\tilde{A}(t)](v) + av$ is uniformly monotone, i.e.

$$\langle [\tilde{A}(t)](v_1) - [\tilde{A}(t)](v_2), v_1 - v_2 \rangle \geq c\|v_1 - v_2\|_V^p - a\|v_1 - v_2\|_H^2$$

with some constant $c > 0$. Then the solution of (7.1) is unique and it depends continuously on $F$ and $u_0$.

Indeed, multiplying the equation (7.1) by $\mathrm{e}^{-at}$, we obtain that $\tilde{u}(t) = \mathrm{e}^{-at}u(t)$ satisfies $\tilde{u}(0) = u_0$ and

$$\tilde{u}(t)' + \mathrm{e}^{-at}[\tilde{A}(t)][\mathrm{e}^{at}\tilde{u}(t)] + a\tilde{u}(t) = \mathrm{e}^{-at}F(t).$$

Applying Remark 7.4 to the operator $\tilde{B}(t)$, defined by

$$[\tilde{B}(t)](v) = \mathrm{e}^{-at}[\tilde{A}(t)][\mathrm{e}^{at}v] + av$$

and to $\tilde{u}$, we obtain the uniqueness of the solution of (7.1) and for $\tilde{u}_j(t) = \mathrm{e}^{-at}u_j(t)$, $\tilde{F}_j(t) = \mathrm{e}^{-at}F_j(t)$ $(j = 1, 2)$ an estimation of the form

$$\|\tilde{u}_1(t) - \tilde{u}_2(t)\|_H^2 + \frac{c}{2}\|\tilde{u}_1 - \tilde{u}_2\|_{L^p(0,T;V)}^p \leq$$

$$\tilde{c}\|\tilde{F}_1 - \tilde{F}_2\|_{L^q(0,T;V^\star)}^q + \|u_{01} - u_{02}\|_H^2.$$

**Remark 7.6.** According to the proof of Theorem 7.1, a subsequence of the Galerkin solutions $(u_m)$ converges weakly in $L^p(0, T; V)$ to a solution $u$ of (7.1). Since the solution of (7.1) is unique, the total sequence $(u_m)$ is also weakly converging to $u$. Further, similarly to the elliptic case, if (7.25) holds, i.e. $\tilde{A}(t)$ is uniformly monotone, then

$$(u_m) \to u \text{ strongly in } L^p(0, T; V).$$

Indeed, assuming that the original sequence does not converge weakly to $u$, by using Cantor's trick, we get a contradiction. Further, by (7.25)

$$c\int_0^T \|u_m(t) - u(t)\|_V^p dt \leq [A(u_m) - A(u), u_m - u] =$$

$$[A(u_m), u_m - u] - [A(u), u_m - u] \to 0$$

by (7.15) and (7.22) since $A$ is pseudomonotone.

# 8 Application to nonlinear parabolic equations

By using the results of Sections 3, one obtains the following applications of Section 7 to nonlinear parabolic equations.

Let $V$ be a closed linear subspace of $W^{1,p}(\Omega)$ (containing $W_0^{1,p}(\Omega)$), $2 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ a bounded domain with "sufficiently smooth" boundary (see, e.g., [1]), $H = L^2(\Omega)$. Then $V \subset H \subset V^\star$ is an evolution triple. We shall consider operators $A : L^p(0,T;V) \to L^q(0,T;V^\star)$, defined by a formula which is analogous to (3.1).

On functions $a_j$ we assume

$(B1)$ Functions $a_j : Q_T \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(j = 1, ..., n)$ satisfy the Carathéodory conditions, i.e. for a.e. fixed $(t,x) \in Q_T = (0,T) \times \Omega$

$$\xi \mapsto a_j(t,x,\xi), \quad \xi \in \mathbb{R}^{n+1} \text{ is continuous}$$

and for each fixed $\xi \in \mathbb{R}^{n+1}$

$$(t,x) \mapsto a_j(t,x,\xi), \quad (t,x) \in Q_T \text{ is measurable.}$$

$(B2)$ There exist a constant $c_1 > 0$ and a function $k_1 \in L^q(Q_T)$ $(1/p + 1/q = 1)$ such that for a.e. $(t,x) \in Q_T$, all $\xi \in \mathbb{R}^{n+1}$

$$|a_j(t,x,\xi)| \le c_1|\xi|^{p-1} + k_1(t,x).$$

$(B3)$ For a.a. $(t,x) \in Q_T$, all $\xi, \xi^\star \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n}[a_j(t,x,\xi) - a_j(t,x,\xi^\star)](\xi_j - \xi_j^\star) \ge 0.$$

$(B4)$ There exist a constant $c_2 > 0$, $k_2 \in L^1(Q_T)$ such that for a.e. $(t,x) \in Q_T$, all $\xi \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n} a_j(t,x,\xi)\xi_j \ge c_2|\xi|^p - k_2(t,x).$$

In this particular case, when $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, for a function $U \in L^p(0,T;V)$ we shall denote $U(t)$ by $u(t,x)$ and instead of $U \in L^p(0,T;V)$ we shall write $u \in L^p(0,T;V)$.

By using the same arguments as in Section 3, one proves

**Theorem 8.1.** *Assume $(B1)$ – $(B4)$. Then the operator $A$, defined by*

$$[A(u), v] = \int_0^T \langle [\tilde{A}(t)][U(t)], v(t)\rangle dt = \tag{8.1}$$

$$\int_0^T \left\{ \int_\Omega \left[ \sum_{j=1}^{n} a_j(t,x,u,Du)D_j v + a_0(t,x,u,Du)v \right] dx \right\} dt, \quad u, v \in L^p(0,T;V)$$

*satisfies the assumptions of Theorem 7.1. Thus for any $F \in L^q(0,T;V^\star)$, $u_0 \in H = L^2(\Omega)$ there is a unique solution $u$ of (7.1) with the operator (8.1).*

Proposition 3.3 implies the following sufficient condition for $(B3)$.

**Proposition 8.2.** *Assume that functions $a_j$ satisfy $(B1)$, further, for a.a. $(t,x) \in Q_T$, the functions $\xi \mapsto a_j(t,x,\xi)$ are continuously differentiable and the matrix*

$$\left( \frac{\partial a_j(t,x,\xi)}{\partial \xi_k} \right)_{j,k=0}^{n}$$

*is positive semidefinite. Then $(B3)$ holds.*

**Proposition 8.3.** *Assume that the assumptions of Proposition 8.2 are fulfilled such that for a.a.* $(t,x) \in Q_T$*, each* $\xi, \eta \in \mathbb{R}^{n+1}$

$$\sum_{j,k=0}^{n} \frac{\partial a_j}{\partial \xi_k}(t,x,\xi)\eta_j\eta_k \geq c_3 \sum_{j=0}^{n} |\xi_j|^{p-2}|\eta_j|^2 \tag{8.2}$$

*with* $p \geq 2$ *and some positive constant* $c_3$*. Then*

$$\sum_{j=0}^{n}[a_j(t,x,\xi) - a_j(t,x,\xi^\star)](\xi_j - \xi_j^\star) \geq \tilde{c}_3 \sum_{j=0}^{n} |\xi_j - \xi_j^\star|^p$$

*with some constant* $\tilde{c}_3 > 0$*. Consequently, the operator* $\tilde{A}(t)$*, defined by (8.1) is uniformly monotone in the sense (7.25) and so the solution of (7.1) is unique and it depends continuously on* $F$ *and* $u_0$ *according to (7.26). Further, due to Remark 7.6 the sequence, constructed by the Galerkin method converges to the solution* $u$ *with respect to the norm of* $L^p(0,T;V)$*.*

**Example 8.4.** A simple example satisfying all the above conditions is the equation

$$D_t u - \triangle_p u - cu|u|^{p-2} = f, \quad c > 0 \text{ is a constant .}$$

(See Example 3.8.)

In the case $V = W_0^{1,p}(\Omega)$ (with bounded $\Omega$) the conditions are satisfied also for the equation

$$D_t u - \triangle_p u = f.$$

## Problems

1. Assume that the functions

$$\alpha_j : Q_T \times \mathbb{R} \to \mathbb{R}, \quad j = 0,1,\dots,n$$

satisfy the Carathéodory conditions and for a.a. $(t,x) \in Q_T$

$$\xi_j \mapsto \alpha_j(t,x,\xi_j) \text{ is monotone nondecreasing,}$$

$$\beta_1|\xi_j|^{p-1} \leq \alpha_j(t,x,\xi_j) \leq \beta_2|\xi_j|^{p-1}, \quad \xi_j \in \mathbb{R}$$

with some positive constants $\beta_1, \beta_2$. Consider the operator

$$[A(u),v] = \int_0^T \left\{ \int_\Omega \left[ \sum_{j=1}^{n} \alpha_j(t,x,D_j u)D_j v + \alpha_0(t,x,u)v \right] dx \right\} dt,$$

$$u, v \in L^p(0,T;V)$$

where $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, $p \geq 2$.

Show that for arbitrary $F \in L^q(0,T : V^\star)$ and $u_0 \in L^2(\Omega)$ there exists a unique solution of problem (7.1).

2. Assume that the functions $\xi_j \mapsto \alpha_j(t,x,\xi_j)$ are continuously differentiable and there exists a positive constant $\beta_3$ such that

$$\frac{\partial \alpha_j}{\partial \xi_j}(t,x,\xi_j) \geq \beta_3 |\xi_j|^{p-2}.$$

By using Remark 7.5 and Proposition 3.4 , show that the solution of the above problem depends continuously on $F$ and $u_0$.

3. Let $\alpha, \beta : Q_T \to \mathbb{R}$ be measurable functions satisfying

$$c_1 \leq \alpha(t,x) \leq c_2, \quad c_1 \leq \beta(t,x) \leq c_2, \quad \text{for almost all } (t,x) \in Q_T$$

with some positive constants $c_1, c_2$. Define operator $A$ by

$$[A(u),v] = \int_0^T \langle [\tilde{A}(t)][u(t)], v(t)\rangle dt =$$

$$\int_0^T \left\{ \int_\Omega [\alpha(t,x) D_j u |Du|^{p-2} D_j v + \beta(t,x) u|u|^{p-2} v] dx \right\} dt,$$

$u, v \in L^p(0,T;V)$ where $V \subset W^{1,p}(\Omega)$ is a closed linear subspace, $p \geq 2$.

By using Theorem 7.1 and Remark 7.5, show that there exists a unique solution of problem (7.1) and it depends continuously on $F$ and $u_0$.

4. Assume that $u \in C^{1,2}(\overline{Q_T})$ is a (classical) solution of (6.1), (6.3) with the boundary condition

$$u(t,x) = g(x), \quad (t,x) \in \Gamma_T$$

where $g(x) = h(x)$ for $x \in \partial\Omega$ and $w_0 \in W^{1,p}(\Omega)$ satisfies $w_0|_{\partial\Omega} = g$. Define the function $u_0$ by $u_0(t,x) = w_0(x)$.

Prove that then the function $\tilde{u} = u - u_0$ satisfies

$$\tilde{u}' + A(\tilde{u} + u_0) = F, \quad \tilde{u} \in W_p^1(0,T;V,H),$$

$$\tilde{u}(0) = h - w_0$$

where $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$, the operator $A$ is defined by (6.6) and $F$ is defined by (6.7).

If $\tilde{u} \in W_p^1(0,T;V,H)$ satisfies the above conditions, $u = \tilde{u} + u_0$ is called a weak solution of the above (classical) initial-boundary value problem.

5. By using Theorem 8.1, show that if the functions $a_j$ satisfy $(B1)$–$(B4)$ then there is a weak solution $u = \tilde{u} + u_0$ of the above problem with nonhomogeneous boundary condition. (See Problem **4**.)

6. Assume that the functions

$$f_\alpha : Q_T \times \mathbb{R} \to \mathbb{R}, \quad |\alpha| \le m$$

satisfy the Carathéodory conditions and for a.a. $(t, x) \in Q_T$

$$\xi_\alpha \mapsto f_\alpha(t, x, \xi_\alpha) \text{ is monotone nondecreasing,}$$

$$\beta_1 |\xi_\alpha|^{p-1} \le |f_\alpha(t, x, \xi_\alpha)| \le \beta_2 |\xi_\alpha|^{p-1}, \quad \xi_\alpha \in \mathbb{R}$$

with some positive constants $\beta_1, \beta_2$. Consider the operator

$$[A(u), v] = \int_0^T \langle [\tilde{A}(t)][u(t)], v(t) \rangle dt =$$

$$\int_0^T \left\{ \int_\Omega \left[ \sum_{|\alpha| \le m} f_\alpha(t, x, D^\alpha u) D^\alpha v \right] dx \right\} dt,$$

$u, v \in L^p(0, T; V)$ where $V \subset W^{m,p}(\Omega)$ is a closed linear subspace, $p \ge 2, m \ge 1$ and for $|\alpha| = 0$, $D^\alpha u = u$ by definition.

Show that for arbitrary $F \in L^q(0, T; V^\star)$ and $u_0 \in L^2(\Omega)$ there exists a unique solution of problem (7.1) with the above operator $A$.

7. Assume the the functions $\xi_\alpha \mapsto f_\alpha(t, x, \xi_\alpha)$ are continuously differentiable and there exists a positive constant $\beta_3$ such that

$$\frac{\partial f_\alpha}{\partial \xi_\alpha}(t, x, \xi_\alpha) \ge \beta_3 |\xi_\alpha|^{p-2}, \quad p \ge 2.$$

By using Proposition 3.4 and Remark 7.5 show that the solution of the above problem depends continuously on $F$ and $u_0$.

# 9 Cauchy problem with pseudomonotone operators

In the proof of Theorem 7.1 we did not use the monotonicity of $\tilde{A}(t)$ directly, it would be sufficient to assume instead of monotonicity and hemicontinuity that $\tilde{A}(t) : V \to V^\star$ is demicontinuous and $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ is pseudomonotone. Moreover, it is sufficient to assume a weaker form of pseudomonotonicity, which will be satisfied for operators of the form (8.1) if the functions $a_j$ satisfy conditions which are analogous to $(\tilde{A}3)$, $(A3')$, respectively.

**Definition 9.1.** *Let $V \subset H \subset V^\star$ be an evolution triple, $p > 1$. A bounded operator $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ is called* pseudomonotone with respect to $W_p^1(0, T; V, H)$ *if*

$$u_k \in W_p^1(0, T; V, H), \quad (u_k) \to u \text{ weakly in } L^p(0, T; V), \tag{9.1}$$

$$(u_k') \to u' \text{ weakly in } L^q(0,T;V^\star), \tag{9.2}$$

$$\limsup_{k\to\infty}[A(u_k), u_k - u] \le 0 \tag{9.3}$$

*imply*

$$\lim_{k\to\infty}[A(u_k), u_k - u] = 0 \text{ and } (A(u_k)) \to A(u) \text{ weakly in } L^q(0,T;V^\star). \tag{9.4}$$

**Theorem 9.2.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$. Assume that for a.a. fixed $t \in [0,T]$, $\tilde{A}(t) : V \to V^\star$ is demicontinuous and bounded such that for all $v \in V$, a.e. $t \in [0,T]$*

$$\|[\tilde{A}(t)](v)\|_{V^\star} \le c_1\|v\|_V^p + k_1(t) \tag{9.5}$$

*with a suitable constant $c_1 > 0$ and $k_1 \in L^q(0,T)$. Further, $\tilde{A}(t)$ is coercive such that for all $v \in V$, a.e. $t \in [0,T]$*

$$\langle[\tilde{A}(t)](v), v\rangle \ge c_2\|v\|_V^p - k_2(t) \tag{9.6}$$

*with some constant $c_2 > 0$, $k_2 \in L^1(0,T)$ and for arbitrary fixed $u, v \in V$, the function*

$$t \mapsto \langle[\tilde{A}(t)](u), v\rangle, \quad t \in [0,T] \text{ is measurable }. \tag{9.7}$$

*Finally, the operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$, defined by $[A(u)](t) = [\tilde{A}(t)][u(t)]$ is pseudomonotone with respect to $W_p^1(0,T;V,H)$.*

*Then for any $F \in X^\star = L^q(0,T;V^\star)$ and $u_0 \in H$ there exists a solution $u$ of (7.1).*

*Proof.* Theorem 9.2 follows by a slight modification of the proof of Theorem 7.1, because we only have to show property (9.2) for a subsequence of the sequence, constructed by Galerkin's method. Clearly it will follow from the fact that the sequence $(u_m')$ is bounded in $L^q(0,T;V^\star)$.

Multiply the equations (7.5) (defining $u_m$ with the initial condition (7.6)) with arbitrary functions $b_{jm} \in L^p(0,T)$ and integrate over $[0,T]$. Then we obtain for the sum of these equations

$$[u_m', w] + [A(u_m), w] = [F, w] \text{ where} \tag{9.8}$$

$$w(t) = \sum_{j=1}^m b_{jm}(t) z_j \text{ and } w \in L^p(0,T;V). \tag{9.9}$$

The equation (9.8) implies

$$|[u_m', w]| \le |[F, w]| + |[A(u_m), w]| \le$$

$$\left[\|F\|_{L^q(0,T;V^\star)} + \|A(u_m)\|_{L^q(0,T;V^\star)}\right]\|w\|_{L^p(0,T;V)} \le \text{const}\|w\|_{L^p(0,T;V)}$$

where the constant is independent of $m$ and $w$.

The functions $w$ of the form (9.9) (for all $m$ and arbitrary $b_{jm} \in L^p(0,T)$) are dense in $L^q(0,T;V^\star)$ because the linear combinations of $z_j$ are dense in $V$, thus

$$|[u'_m, w]| \leq \text{const}\|w\|_{L^p(0,T;V)}$$

holds for all $w \in L^p(0,T;V)$ (with a constant, not depending on $w$). Thus we obtained that $(u'_m)$ is bounded with respect to the norm of $L^q(0,T;V^\star)$, the dual space of $L^p(0,T;V)$, which completes the proof of Theorem 9.2. □

Now we shall formulate a generalization of Theorem 9.2. Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$. Define operator $L$ as follows:

$$Lu = u', \quad u \in D(L) = \{u \in W_p^1(0,T;V,H) : u(0) = 0\} \tag{9.10}$$

One can show that $L$ is a closed, linear, densely defined operator from $L^p(0,T;V)$ into $L^q(0,T;V^\star)$, which is monotone by Remark 6.7 since

$$[Lu, u] = \int_0^T \langle u'(t), u(t)\rangle dt = \frac{1}{2}\|u(T)\|_H^2 \geq 0.$$

Further, $L$ is "maximal monotone", which means that there is no proper monotone extension of it. (For the proof see, e.g., [93].)

Another example of a closed, linear, densely defined maximal monotone operator is (see, Theorem 13.2):

$$\tilde{L}u = u', \quad u \in D(\tilde{L}) = \{u \in W_p^1(0,T;V,H) : u(T) = u(0)\}. \tag{9.11}$$

**Definition 9.3.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$. Denote by $L$ a closed, linear, densely defined, maximal monotone operator from $L^p(0,T;V)$ into $L^q(0,T;V^\star)$. A bounded operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is called* pseudomonotone with respect to $D(L)$ *if*

$$u_k, u \in D(L), \quad (u_k) \to u \textit{ weakly in } L^p(0,T;V),$$

$$(Lu_k) \to Lu \textit{ weakly in } L^q(0,T;V^\star), \quad \limsup_{k\to\infty}[A(u_k), u_k - u] \leq 0$$

*imply*

$$\lim_{k\to\infty}[A(u_k), u_k - u] = 0 \textit{ and } (A(u_k)) \to A(u) \textit{ weakly in } L^q(0,T;V^\star).$$

**Theorem 9.4.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$. Denote by $L$ a closed, linear, densely defined, maximal monotone operator from $L^p(0,T;V)$ into $L^q(0,T;V^\star)$. Assume that $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, pseudomonotone with respect to $D(L)$ and coercive.*

*Then for all $F \in L^q(0,T;V^\star)$ there exists a solution $u \in D(L)$ of*

$$Lu + A(u) = F.$$

For the proof see, e.g., [8].

It is important that in Theorem 9.4 $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is not assumed to have the form

$$[A(u)](t) = [\tilde{A}(t)][u(t)], \tag{9.12}$$

i.e. $[A(u)](t)$ may depend not only on $u(t)$, thus the above theorem can be applied to “functional parabolic equations”. (See some examples in Section 10.)

**Remark 9.5.** Applying Theorem 9.4 with operator $\tilde{L}$, defined by (9.11) and operator $A$, defined by (9.12), one obtains existence of $T$-periodic solutions, see Section 13.

Now consider the particular case when $Lu = u'$ and $D(L)$ is defined by (9.10). We generalize the existence theorem to the case of nonhomogeneous initial condition.

**Theorem 9.6.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$ and let $L$ be defined by (9.10). Assume that $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, pseudomonotone with respect to $W_p^1(0,T;V,H)$ and coercive such that for arbitrary constant $c > 0$*

$$\lim_{\|u\|\to\infty} \frac{\int_0^T \langle [A(u)](t), u(t)\rangle dt - c\|A(u)\|_{L^q(0,T;V^\star)}}{\|u\|_{L^p(0,T;V)}} = +\infty.$$

*Then for all $F \in L^q(0,T;V^\star)$, $u_0 \in H$ there exists a solution $u \in W_p^1(0,T;V,H)$ of*

$$u' + A(u) = F, \quad u(0) = u_0. \tag{9.13}$$

*Proof.* If $u_0 \in V$, one can reduce problem (9.13) to the case $u_0 = 0$ as follows. By using the notations $u_0(t) = u_0$, $t \in [0,T]$, $\tilde{u} = u - u_0$, problem (9.13) is equivalent to the problem

$$\tilde{u}' + A(\tilde{u} + u_0) = F, \quad \tilde{u}(0) = 0. \tag{9.14}$$

Clearly, the operator $\tilde{u} \mapsto A(\tilde{u} + u_0)$ is demicontinuous, bounded and pseudomonotone with respect to $D(L)$. Further, it is coercive because

$$\frac{[A(\tilde{u}+u_0), \tilde{u}]}{\|\tilde{u}\|_{L^p(0,T;V)}} = \frac{[A(\tilde{u}+u_0), \tilde{u}+u_0] - [A(\tilde{u}+u_0), u_0]}{\|\tilde{u}\|_{L^p(0,T;V)}} \geq$$

$$\frac{[A(\tilde{u}+u_0), \tilde{u}+u_0] - \|u_0\|_{L^p(0,T;V)}\|A(\tilde{u}+u_0)\|_{L^q(0,T;V^\star)}}{\|\tilde{u}+u_0\|_{L^p(0,T;V)}} \cdot$$

$$\frac{\|\tilde{u}+u_0\|_{L^p(0,T;V)}}{\|\tilde{u}\|_{L^p(0,T;V)}} \to +\infty$$

if $\|\tilde{u}\|_{L^p(0,T;V)} \to \infty$ since then $\|\tilde{u} + u_0\|_{L^p(0,T;V)} \to \infty$. Thus for any $u_0 \in H$ there is a solution $\tilde{u}$ of (9.13) by Theorem 9.4.

Now let $u_0 \in H$ arbitrary element. Since $V$ is dense in $H$, there is a sequence of $u_{n0} \in V$, converging to $u_0$ in $H$. According to the first part of the proof, there is a solution $u_n \in W_p^1(0,T;V,H)$ of

$$u_m' + A(u_m) = F, \quad u_m(0) = u_{m0}.$$

By using the arguments of the proof of Theorem 7.1, we obtain that there is a subsequence of $(u_m)$ which converges weakly in $L^p(0,T;V)$ to a solution of (9.13). $\square$

# 10 Parabolic equations and functional equations

## Parabolic differential equations

Here we shall apply the results of Section 9 to the case when $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, $2 \le p < \infty$, $\Omega \subset \mathbb{R}^n$ is a bounded domain (with sufficiently smooth boundary), $H = L^2(\Omega)$. First we shall consider operators $A$ of the form (8.1), but instead of $(B3)$, with weaker assumptions, which are analogous to assumptions $(\tilde{A}3)$, $(A3')$, respectively, considered in the nonlinear elliptic case. It will be proved that $A$ is pseudomonotone with respect to $W_p^1(0,T;V,H)$, by using the following compact imbedding theorem.

**Theorem 10.1.** *Let $V \subset H \subset V^\star$ be an evolution triple, $B$ a Banach space satisfying*

$$V \subset B \subset V^\star, \text{ the imbedding } V \subset B \text{ is compact, } B \subset V^\star \text{ is continuous }. \tag{10.1}$$

*Then for any $1 < p < \infty$, the imbedding*

$$W_p^1(0,T;V,H) \subset L^p(0,T;B)$$

*is compact.*

In the proof of Theorem 10.1 we shall use

**Lemma 10.2.** *Assume (10.1). Then for arbitrary $\eta > 0$ there exists a constant $c_\eta > 0$ such that for all $v \in V$*

$$\|v\|_B \le \eta\|v\|_V + c_\eta\|v\|_{V^\star}. \tag{10.2}$$

*Proof.* Assume that (10.2) does not hold, then there exists $\eta > 0$ and sequences $(c_k), (v_k)$, $c_k \in \mathbb{R}$, $v_k \in V$, satisfying

$$\|v_k\|_B > \eta\|v_k\|_V + c_k\|v_k\|_{V^\star}, \quad \lim_{k\to\infty} c_k = +\infty. \tag{10.3}$$

Then for $w_k = v_k/\|v_k\|_V$ we have

$$\|w_k\|_B > \eta + c_k\|w_k\|_{V^\star}, \quad \|w_k\|_B = \frac{\|v_k\|_B}{\|v_k\|_V} \le \text{const} \tag{10.4}$$

because the imbedding $V \subset B$ is continuous. Thus (10.3), (10.4) imply

$$\lim_{k\to\infty} \|w_k\|_{V^\star} = 0. \tag{10.5}$$

Further, since $\|w_k\|_V = 1$ and the imbedding $V \subset B$ is compact, there is a subsequence $(w_{k_l})$ of $(w_k)$ which is convergent in $B$. Due to (10.5) the limit in $B$ must be 0, i.e.

$$\lim_{l\to\infty} \|w_{k_l}\|_B = 0$$

which is impossible because of (10.4). □

*Proof of Theorem 10.1.* Let $(v_k)$ be a bounded sequence in $W_p^1(0,T;V,H)$. We have to show that a subsequence is convergent in $L^p(0,T;B)$. First observe that as $W_p^1(0,T;V,H)$ is a reflexive Banach space ($V$, $V^\star$ are reflexive thus $L^p(0,T;V)$, $L^q(0,T;V^\star)$ are reflexive), thus there are $v \in W_p^1(0,T;V,H)$ and a subsequence of $(v_k)$, again denoted by $(v_k)$ such that

$$(v_k) \to v \text{ weakly in } W_p^1(0,T;V,H), \text{ thus} \tag{10.6}$$

$$(v_k - v) \to 0 \text{ weakly in } W_p^1(0,T;V,H).$$

To prove our theorem, we have to show that

$$(v_k - v) \to 0 \text{ in } L^p(0,T;B). \tag{10.7}$$

Introduce the notation $\tilde{v}_k = v_k - v$, due to (10.6) we have

$$(\tilde{v}_k) \to 0 \text{ weakly in } W_p^1(0,T;V,H), \quad \|\tilde{v}_k\|_{W_p^1(0,T;V,H)} \le c^\star \tag{10.8}$$

with some constant $c^\star > 0$. We prove that

$$(\tilde{v}_k) \to 0 \text{ in } L^p(0,T;B). \tag{10.9}$$

By Lemma 10.2 for arbitrary $\eta > 0$ there exists $c_\eta > 0$ such that

$$\|\tilde{v}_k\|_B \le \eta\|\tilde{v}_k\|_V + c_\eta\|\tilde{v}_k\|_{V^\star}$$

which implies

$$\|\tilde{v}_k\|_{L^p(0,T;B)} \le \eta\|\tilde{v}_k\|_{L^p(0,T;V)} + c_\eta\|\tilde{v}_n k\|_{L^p(0,T;V^\star)} \le \tag{10.10}$$

$$c^\star\eta + c_\eta\|\tilde{v}_k\|_{L^p(0,T;V^\star)}.$$

Since (10.10) holds for arbitrary $\eta > 0$, we shall obtain (10.9) by showing

$$(\tilde{v}_k) \to 0 \text{ in } L^p(0,T;V^\star). \tag{10.11}$$

The convergence (10.11) will follow from Lebesgue's dominated convergence theorem, if we show that for almost all $s \in [0, T]$,

$$\tilde{v}_k(s) \to 0 \text{ with respect to the norm of } V^\star. \tag{10.12}$$

Indeed, for a.a. $s \in [0, T]$, $k \in \mathbb{N}$

$$\|\tilde{v}_k(s)\|_{V^\star} \leq \text{const}$$

since $(\tilde{v}_k)$ is bounded in $W_p^1(0, T; V, H)$ and by Theorem 6.6 $W_p^1(0, T; V, H)$ is continuously imbedded into $C([0, T]; H)$, hence into $C([0, T]; V^\star)$, too.

Now we prove (10.12). For simplicity, consider the case $s = 0$, the general case can be treated similarly. Define functions $u_k$ by

$$u_k(t) = \tilde{v}_k(\lambda t), \quad t \in [0, T] \tag{10.13}$$

where the constant $\lambda \in (0, 1)$ will be chosen later. By the definition (10.13) $u_k(0) = \tilde{v}_k(0)$, and as $(\tilde{v}_k)$ is bounded in $W_p^1(0, T; V, H)$, we obtain inequalities

$$\|u_k\|_{L^p(0,T;V)} \leq d_1 \lambda^{-1/p}, \quad \|u_k'\|_{L^q(0,T;V^\star)} \leq d_2 \lambda^{1/p} \tag{10.14}$$

with some constants $d_1, d_2 > 0$, not depending on $\lambda$. Let $\varphi \in C^1[0, T]$ be a function with the properties $\varphi(0) = -1$, $\varphi(T) = 0$. Then

$$u_k(0) = \int_0^T (\varphi u_k)' dt = \int_0^T \varphi u_k' dt + \int_0^T \varphi' u_k dt = \beta_k + \gamma_k \tag{10.15}$$

whence by (10.14)

$$\|\tilde{v}_k(0)\|_{V^\star} = \|u_k(0)\|_{V^\star} \leq \|\beta_k\|_{V^\star} + \|\gamma_k\|_{V^\star} \leq \tag{10.16}$$

$$d_3 \lambda^{1/p} + \|\gamma_k\|_{V^\star}.$$

The number $\lambda \in (0, 1)$ can be chosen such that the first term in the right hand side of (10.16) is arbitrary small for all $n \in \mathbb{N}$. Therefore, we shall obtain (10.12) for $s = 0$ if we show that

$$\|\gamma_k\| \to 0 \text{ in } V^\star. \tag{10.17}$$

According to (10.8) $(\tilde{v}_k) \to 0$ weakly in $W_p^1(0, T; V, H)$, thus $(\tilde{v}_k) \to 0$ and so $(u_k) \to 0$ weakly in $L^p(0, T; V)$ for arbitrary fixed $\lambda \in (0, 1)$. Consequently, by the definition (10.15) of $\gamma_k$,

$$(\gamma_k) \to 0 \text{ weakly in } V. \tag{10.18}$$

Since the imbedding $V \subset V^\star$ is compact, (10.18) implies (10.17) which completes the proof of Theorem 10.1. $\square$

Now instead of $(B3)$ we formulate a weaker assumption on functions $a_j$, defining operators $\tilde{A}(t)$ and $A$ in (8.1) which will imply with $(B1)$, $(B2)$, $(B4)$ that $\tilde{A}(t)$ satisfies assumptions of Theorem 9.2.

As in Section 8, let $V$ be a closed linear subspace of $W^{1,p}(\Omega)$, $2 \le p < \infty$, $\Omega \subset \mathbb{R}^n$ a bounded domain (with sufficiently smooth boundary), $H = L^2(\Omega)$. Instead of $(B3)$ we assume on functions $a_j : Q_T \times \mathbb{R}^n \to \mathbb{R}$

$(\tilde{B}3)$ There exists a constant $\tilde{c}_2 > 0$ such that for a.e. $(t,x) \in Q_T$, all $\eta \in \mathbb{R}$, $\zeta, \zeta^\star \in \mathbb{R}^n$

$$\sum_{j=1}^n [a_j(t,x,\eta,\zeta) - a_j(t,x,\eta,\zeta^\star)](\zeta_j - \zeta_j^\star) \ge \tilde{c}_2 |\zeta - \zeta^\star|^p.$$

**Remark 10.3.** Assumption $(\tilde{B}3)$ is analogous to $(\tilde{A}3)$ in Section 4.

**Theorem 10.4.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\partial\Omega$ is sufficiently smooth and $(B1)$, $(B2)$, $(\tilde{B}3)$, $(B4)$ hold. Then operator $A$ of the form (8.1) satisfies all the conditions of Theorem 9.2.*

*Proof.* All the conditions easily follow from the above conditions (see Theorem 8.1), we only have to show that $A$ is pseudomonotone with respect to $W_p^1(0,T;V,H)$. Assume that

$$(u_k) \to u \text{ weakly in } L^p(0,T;V), \tag{10.19}$$

$$(u_k') \to u' \text{ weakly in } L^q(0,T;V^\star) \text{ and}$$
$$\limsup_{k\to\infty} [A(u_k), u_k - u] \le 0. \tag{10.20}$$

Since $W^{1,p}(\Omega)$ is compactly imbedded into $L^p(\Omega)$ (for bounded $\Omega$ with sufficiently smooth boundary, see Theorem 4.1), by Theorem 10.1 there is a subsequence of $(u_k)$, again denoted by $(u_k)$, for simplicity, such that

$$(u_k) \to u \text{ in } L^p(Q_T) \text{ and a.e. in } Q_T. \tag{10.21}$$

The remaining part of the proof is similar to that of Theorem 4.6. Since $(D_j u_k)$ is bounded in $L^p(Q_T)$, we may assume (on the subsequence) that

$$(D_j u_k) \to D_j u \text{ weakly in } L^p(Q_T), \quad j = 1, ..., n. \tag{10.22}$$

Further,

$$[A(u_k), u_k - u] = \int_{Q_T} a_0(t,x,u_k,Du_k)(u_k - u)dtdx + \tag{10.23}$$

$$\sum_{j=1}^n \int_{Q_T} [a_j(t,x,u_k,Du_k) - a_j(t,x,u_k,Du)](D_j u_k - D_j u)dtdx +$$

$$\sum_{j=1}^n \int_{Q_T} a_j(t,x,u_k,Du)(D_j u_k - D_j u)dtdx.$$

The first term on the right-hand side of (10.23) tends to 0 by (10.21) and Hölder's inequality, because the multipliers of $(u_k - u)$ are bounded in $L^q(Q_T)$ (by $(B2)$). Further, the third term on the right-hand side converges to 0, too, because (10.21), $(B1)$, $(B2)$ and Vitali's convergence theorem imply that

$$a_j(t, x, u_k, Du) \to a_j(t, x, u, Du) \text{ in } L^q(Q_T).$$

Consequently, (10.20), (10.23) imply

$$\limsup_{k\to\infty} \sum_{j=1}^{n} \int_{Q_T} [a_j(t, x, u_k, Du_k) - a_j(t, x, u_k, Du)](D_j u_k - D_j u)dtdx \leq 0. \tag{10.24}$$

>From $(\tilde{B}3)$, (10.24) we obtain

$$\lim_{k\to\infty} \int_{Q_T} |Du_k - Du|^p dtdx = 0 \tag{10.25}$$

and (for a subsequence)

$$(Du_k) \to Du \text{ a.e. in } Q_T. \tag{10.26}$$

Therefore, by $(B1)$, $(B2)$, (10.25), (10.21), (10.26) and Vitali's theorem (Theorem 4.3)

$$a_j(t, x, u_k, Du_k) \to a_j(t, x, u, Du) \text{ in } L^q(Q_T), \quad j = 0, 1, ..., n.$$

Thus by Hölder's inequality

$$(A(u_k)) \to A(u) \text{ weakly in } L^q(0, T; V^\star). \tag{10.27}$$

Finally, from (10.21), (10.23), (10.25) and $(B2)$ one gets

$$\lim_{k\to\infty} [A(u_k), u_k - u] = 0. \tag{10.28}$$

Since (10.27), (10.28) hold for a subsequence of $(u_k)$, by using Cantor's trick, we obtain (10.27), (10.28) for the original sequence. $\square$

**Remark 10.5.** According to the proof of the above theorem, operator $A$ belongs to the class $(S)_+$ and it is demicontinuous.

Now we formulate assumptions $(B3')$, $(B4')$, which are analogous to $(A3')$, $(A4')$ in Section 4 which will also imply with $(B1)$, $(B2)$ that the conditions of Theorem 9.2 hold.

$(B3')$ For a.e. $(t, x) \in Q_T$, all $\eta \in \mathbb{R}$, $\zeta, \zeta^\star \in \mathbb{R}^n$, $\zeta = (\xi_1, ..., \xi_n) \neq \zeta^\star = (\xi_1^\star, ..., \xi_n^\star)$ we have

$$\sum_{j=1}^{n} [a_j(t, x, \eta, \zeta) - a_j(t, x, \eta, \zeta^\star)](\xi_j - \xi_j^\star) > 0.$$

$(B4')$ There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(Q_T)$ such that for a.e. $(t,x) \in Q_T$, all $\xi = (\eta, \zeta) \in \mathbb{R}^{n+1}$ ($\eta = \xi_0 \in \mathbb{R}$, $\zeta \in \mathbb{R}^n$)

$$\sum_{j=0}^{n} a_j(t,x,\eta,\zeta)\xi_j \geq c_2|\zeta|^p - k_2(t,x).$$

**Theorem 10.6.** *Assume (B1), (B2), (B3′), (B4). Then the operator A, defined by (8.1) satisfies the conditions of Theorem 9.2. Thus, for any* $F \in L^q(0,T;V^\star)$, $u_0 \in H = L^2(\Omega)$ *there is a solution of (7.1) with the operator (8.1).*

*In the case when* $V = W_0^{1,p}(\Omega)$, *instead of (B4) it is sufficient to assume (B4′), because then (B4′) implies coercivity. (See Remarks 3.9, 4.11.)*

*Proof.* As in Section 3, one proves that $(B1)$, $(B2)$ imply (9.5), (9.7) and $(B4)$ implies (9.6). Further, by Theorem 4.9, the operator $\tilde{A}(t) : V \to V^\star$ (defined in (8.1)) is pseudomonotone for a.a. $t \in [0,T]$ (since $(B3')$, $(B4')$ imply: $(A3')$, $(A4')$ hold for a.a. fixed $t \in [0,T]$). Thus, for a.a. $t \in [0,T]$, the bounded operator $A$ is demicontinuous (see Proposition 2.7).

Finally, we have to prove that $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ (defined by (8.1)) is pesudomonotone with respect to $W_p^1(0,T;V,H)$. The proof of this fact is similar to that of Theorem 4.9 (elliptic case) and we use only $(B4')$ instead of $(B4)$.

According to Definition 9.1, assume (9.1) – (9.3), i.e.

$$u_k \in W_p^1(0,T;V,H), \quad (u_k) \to u \text{ weakly in } L^p(0,T;V), \tag{10.29}$$

$$(u_k') \to u \text{ weakly in } L^q(0,T;V^\star), \tag{10.30}$$

$$\limsup_{k\to\infty}[A(u_k), u_k - u] \leq 0. \tag{10.31}$$

We have to show (9.4), i.e.

$$\lim_{k\to\infty}[A(u_k), u_k - u] = 0 \text{ and } (A(u_k)) \to A(u) \text{ weakly in } L^q(0,T;V^\star). \tag{10.32}$$

Since $\Omega \subset \mathbb{R}^n$ is bounded and $\partial\Omega$ is sufficiently smooth, by Theorem 4.1 $V$ is compactly imbedded into $L^p(\Omega)$ and thus by Theorem 10.1 the imbedding

$$W_p^1(0,T;V,H) \subset L^p(0,T;L^p(\Omega)) = L^p(Q_T)$$

is compact. Hence, by (10.29), (10.30) there is a subsequence of $(u_k)$, again denoted by $(u_k)$ (for simplicity) with the properties

$$(u_k) \to u \text{ in } L^p(Q_T) \text{ and a.e. in } Q_T. \tag{10.33}$$

Then the proof of (10.32) is almost the same as that of (4.20) in the proof of Theorem 4.9. Introduce the notation

$$p_k(t,x) = \sum_{j=1}^{n}[a_j(t,x,u_k,Du_k) - a_j(t,x,u,Du)](D_j u_k - D_j u) +$$

$$[a_0(t,x,u_k,Du_k) - a_0(t,x,u,Du)](u_k - u)$$

which is similar to the formula (4.23) of $p_k(x)$. Then

$$[A(u_k) - A(u), u_k - u] = \int_{Q_T} p_k(t,x)dtdx$$

and by (10.29), (10.30) (10.31) we have

$$\limsup_{k\to\infty} \int_{Q_T} p_k(t,x)dtdx \le 0.$$

By using the arguments of the proof of Theorem 4.9, we find

$$\lim_{k\to\infty} \int_{Q_T} p_k(t,x)dtdx = 0 \text{ and} \tag{10.34}$$

$$(p_k) \to 0 \text{ a.e. in } Q_T. \tag{10.35}$$

The equality (10.34) directly implies the first part of (10.32). Further, (10.35), (10.33) and $(B3')$ imply (as in the proof of Theorem 4.9)

$$(Du_k) \to Du \text{ a.e. in } Q_T. \tag{10.36}$$

Finally, by using (10.33), (10.36), $(B1)$, $(B2)$ and Vitali's theorem (Theorem 4.3) we obtain the second part of (10.32) which completes the proof of Theorem 10.6. □

**Remark 10.7.** One can formulate and prove a generalization of Theorem 10.6 to the case when $\tilde{A}(t)$ is a $2m$ order nonlinear elliptic operator which is analogous to (4.37). (See Remark 4.12.)

## Functional parabolic equations

Now we shall show some applications of Theorem 9.4 which is a generalization of Theorem 9.2. In Theorem 9.4 $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is such that $[A(u)](t)$ is depending not only on $u(t)$, thus also "functional parabolic equations" (e.g. equations with delay) can be treated. The following theorem will be a generalization of Theorem 10.4 to functional parabolic equations with nonlinear and "non-local" third boundary conditions.

Let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace ($2 \le p < \infty$, $\Omega \subset \mathbb{R}^n$ a bounded domain with sufficiently smooth boundary), $H = L^2(\Omega)$. We shall consider operators of the following form.

**Definition 10.8.** *Define operator $A$ by*

$$[A(u), v] = \tag{10.37}$$

$$\int_{Q_T} \left\{ \sum_{j=1}^n a_j(t,x,u(x),Du(x);u)D_j v(x) + a_0(t,x,u(x),Du(x);u)v(x) \right\} dtdx +$$

$$\int_0^T \left\{ \int_{\partial\Omega} h(t,x;u) v d\sigma_x \right\} dt, \quad u, v \in L^p(0,T;V).$$

Assume that the following conditions are fulfilled.

$(C1)$ The functions $a_j : Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R}$ $(j = 0,1,...,n)$ satisfy the Carathéodory conditions for arbitrary fixed $u \in L^p(0,T;V)$ and $h : (0,T) \times \partial\Omega \times L^p(0,T;V) \to \mathbb{R}$ is measurable for each fixed $u \in L^p(0,T;V)$.

$(C2)$ There exist (nonlinear) operators $g_1 : L^p(0,T;V) \to \mathbb{R}^+$ and $k_1 : L^p(0,T;V) \to L^q(Q_T)$ such that

$$|a_j(t,x,\eta,\zeta;u)| \leq g_1(u)[1 + |\eta|^{p-1} + |\zeta|^{p-1}] + [k_1(u)](t,x)$$

for a.e. $(t,x) \in Q_T$, each $(\eta,\zeta) \in \mathbb{R}^{n+1}$, $u \in L^p(0,T;V)$ where

$$|g_1(u)| \leq \text{const} \left[1 + \|u\|_{L^p(0,T;V)}\right]^{-\sigma^\star},$$

$$\|k_1(u)\|_{L^q(Q_T)} \leq \text{const} \left[1 + \|u\|_{L^p(0,T;V)}\right]^{\sigma}$$

and the constants $\sigma^\star, \sigma$ satisfy $0 \leq \sigma^\star < p-1$, $0 < \sigma < p - \sigma^\star$.

$(C3)$ The inequality

$$\sum_{j=1}^n [a_j(t,x,\eta,\zeta;u) - a_j(t,x,\eta,\zeta^\star;u)](\xi_j - \xi_j^\star) \geq [g_2(u)](t)|\zeta - \zeta^\star|^p$$

holds where the operator $g_2$ satisfies

$$[g_2(u)](t) \geq c^\star \left[1 + \|u\|_{L^p(0,t;V)}\right]^{-\sigma^\star}$$

and $c^\star$ is some positive constant.

$(C4)$ The inequality

$$\sum_{j=1}^n a_j(t,x,\eta,\zeta;u)\xi_j \geq [g_2(u)](t)[1 + |\eta|^p + |\zeta|^p] - [k_2(u)](t,x) \tag{10.38}$$

holds where $k_2(u) \in L^1(Q_T)$ satisfies for all $t \in [0,T]$

$$\|k_2(u)\|_{L^1(Q_t)} \leq \text{const} \left[1 + \|u\|_{L^p(0,t;V)}\right]^{\sigma}, \quad u \in L^p(0,T;V). \tag{10.39}$$

Further, for all $t \in [0,T]$, $u \in L^p(0,T;V)$

$$\|h(t,x;u)\|_{L^q((0,t)\times\partial\Omega)} \leq \text{const} \left[1 + \|u\|_{L^p(0,t;V)}\right]^{\sigma-1}. \tag{10.40}$$

(In the case $V = W_0^{1,p}(\Omega)$ $h$ is considered to be identically 0.)

$(C5)$ There exists $\delta > 0$ satisfying $\delta < 1 - 1/p$ such that if $(u_k) \to u$ weakly in $L^p(0,T;V)$ and strongly in $L^p(0,T;W^{1-\delta,p}(\Omega))$, $(\eta^k) \to \eta$ in $\mathbb{R}$, $(\zeta^k) \to \zeta$ in $\mathbb{R}^n$ then for a.a. $(t,x) \in Q_T$, $j = 0,1,...,n$

$$\lim_{k\to\infty} a_j(t,x,\eta^k,\zeta^k;u_k) = a_j(t,x,\eta,\zeta;u)$$

for a subsequence and for a.a. $t \in (0,T)$, $x \in \partial\Omega$

$$\lim_{k\to\infty} h(t,x;u_k) = h(t,x;u)$$

for a suitable subsequence.

**Theorem 10.9.** *Assume $(C1)$ – $(C5)$. Then $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, pseudomonotone with respect to $W_p^1(0,T;V,H)$ and coercive in the sense of Theorem 9.6. Thus for any $F \in L^q(0,T;V^\star)$, $u_0 \in L^2(\Omega)$ there exists $u \in W_p^1(0,T;V,H)$ satisfying*

$$u' + A(u) = F, \quad u(0) = u_0. \tag{10.41}$$

*Proof.* Clearly, $(C1)$, $(C2)$ and (10.40) imply that $A$ is bounded, because the trace operator $W^{1-\delta,p}(\Omega) \to L^p(\partial\Omega)$ is bounded if $\delta + 1/p < 1$ (see Remark 4.2) and so by Hölder's inequality for all $v \in V$

$$\Big| \int_{\partial\Omega} h(t,x;u) v \, d\sigma_x \Big| \le \left[ \int_{\partial\Omega} |h(t,x;u)|^q d\sigma_x \right]^{1/q} \cdot \text{const} \|v\|_{W^{1-\delta,p}(\Omega)},$$

hence by (10.40)

$$\Big| \int_0^T \left[ \int_{\partial\Omega} h(t,x;u) v \, d\sigma_x \right] dt \Big| \le \tag{10.42}$$

$$\text{const} \left[ 1 + \|u\|_{L^p(0,T;V)} \right]^{\sigma-1} \|v\|_{L^p(0,T;V)}.$$

Further, by using $(C1)$, $(C2)$, $(C5)$, (10.40), Hölder's inequality and Vitali's theorem (Theorem 4.3) one obtains that $A$ is demicontinuous. Assumptions $(C2)$, $(C4)$ imply that $A$ is coercive in the sense of Theorem 9.6, because (for sufficiently large $\|u\|_{L^p(0,T;V)}$)

$$\frac{[A(u),u] - c\|A(u)\|_{L^q(0,T;V^\star)}}{\|u\|_{L^p(0,T;V)}} \ge \frac{c^\star}{2} \left[ 1 + \|u\|_{L^p(0,T;V)} \right]^{p-1-\sigma^\star} -$$

$$\text{const} \left[ 1 + \|u\|_{L^p(0,T;V)} \right]^{\sigma-1} \to +\infty$$

as $\|u\|_{L^p(0,T;V)} \to \infty$ since $p - \sigma^\star > \sigma$.

Now we show (similarly to the proof of Theorem 4.15) that $A$ is pseudomonotone with respect to $W_p^1(0,T;V,H)$. Assume that

$$(u_k) \to u \text{ weakly in } L^p(0,T;V), \tag{10.43}$$

$$(u_k') \to u' \text{ weakly in } L^q(0,T;V^\star) \text{ and} \tag{10.44}$$

$$\limsup_{k\to\infty} [A(u_k), u_k - u] \le 0. \tag{10.45}$$

Since $W^{1,p}(\Omega)$ is compactly imbedded into $W^{1-\delta,p}(\Omega)$ (for bounded $\Omega$ with "sufficiently good" boundary, see Remark 4.2), by Theorem 10.1 there is a subsequence of $(u_k)$, again denoted by $(u_k)$, for simplicity, such that

$$(u_k) \to u \text{ in } L^p(0,T;W^{1-\delta,p}(\Omega)) \text{ and a.e. in } Q_T. \tag{10.46}$$

Further, since the trace operator $W^{1-\delta,p}(\Omega) \to L^p(\partial\Omega)$ is continuous, the sequence of functions

$$(t,x) \mapsto u_k(t,x), \quad (t,x) \in (0,T) \times \partial\Omega \text{ converges to} \tag{10.47}$$

$$(t,x) \mapsto u(t,x), \quad (t,x) \in (0,T) \times \partial\Omega \text{ in } L^p((0,T) \times \partial\Omega).$$

Since $(D_j u_k)$ is bounded in $L^p(Q_T)$, we may assume (on the subsequence) that

$$(D_j u_k) \to D_j u \text{ weakly in } L^p(Q_T), \quad j = 1, ..., n. \tag{10.48}$$

Further,

$$[A(u_k), u_k - u] = \int_{Q_T} a_0(t,x,u_k,Du_k;u_k)(u_k - u)dtdx+ \tag{10.49}$$

$$\sum_{j=1}^n \int_{Q_T} [a_j(t,x,u_k,Du_k;u_k) - a_j(t,x,u_k,Du;u_k)](D_j u_k - D_j u)dtdx+$$

$$\sum_{j=1}^n \int_{Q_T} a_j(t,x,u_k,Du;u_k)(D_j u_k - D_j u)dtdx + \int_{(0,T)\times\partial\Omega} h(t,x;u_k)(u_k - u)dtd\sigma_x.$$

The first term on the right-hand side of (10.49) tends to 0 by (10.46) and Hölder's inequality, because the multiplier of $(u_k - u)$ is bounded in $L^q(Q_T)$. Further, the third term on the right-hand side converges to 0, too, by (10.48) because (10.43), (10.46), $(C1)$, $(C2)$, $(C5)$ and Vitali's theorem imply that

$$a_j(t,x,u_k,Du;u_k) \to a_j(t,x,u,Du;u) \text{ in } L^q(Q_T).$$

The last term on the right-hand side of (10.49) tends to 0, too, by Hölder's inequality, (10.47) and (10.40).

Consequently, (10.45), (10.49) imply

$$\limsup_{k\to\infty} \sum_{j=1}^n \int_{Q_T} [a_j(t,x,u_k,Du_k;u_k) - a_j(t,x,u_k,Du;u_k)](D_j u_k - D_j u)dtdx \le 0. \tag{10.50}$$

Since $(u_k)$ is bounded in $L^p(0,T;V)$, from $(C3)$, (10.50) we obtain

$$\lim_{k\to\infty} \int_{Q_T} |Du_k - Du|^p dtdx = 0 \tag{10.51}$$

and (for a subsequence)

$$(Du_k) \to Du \text{ a.e. in } Q_T. \tag{10.52}$$

Therefore, by $(C1)$, $(C2)$, $(C5)$, (10.40), (10.43), (10.46), (10.52) and Vitali's theorem (Theorem 4.3)

$$a_j(t,x,u_k,Du_k;u_k) \to a_j(t,x,u,Du;u) \text{ in } L^q(Q_T), \quad j=0,1,...,n,$$

$$h(t,x;u_k) \to h(t,x;u) \text{ in } L^q((0,T)\times\partial\Omega).$$

Thus by Hölder's inequality and Vitali's theorem

$$(A(u_k)) \to A(u) \text{ weakly in } L^q(0,T;V^\star). \tag{10.53}$$

Finally, from (10.46), (10.49), (10.51) and $(C2)$ one gets

$$\lim_{k\to\infty}[A(u_k), u_k - u] = 0. \tag{10.54}$$

Since (10.53), (10.54) hold for a subsequence of $(u_k)$, by Cantor's trick we obtain (10.53), (10.54) for the original sequence.

So we have proved that $A$ is bounded, demicontinuous, pseudomonotone with respect to $W_p^1(0,T;V,H)$ and coercive, thus Theorem 9.6 implies Theorem 10.9. □

**Remark 10.10.** According to the proof of Theorem 10.9, $(C1)$ – $(C5)$ imply that $A$ belongs to the class $(S)_+$ with respect to $W_p^1(0,T;V,H)$, i.e.

$$(u_k) \to u \text{ weakly in } L^p(0,T;V), \quad (u_k') \to u' \text{ weakly in } L^q(0,T;V^\star),$$

$$\limsup_{k\to\infty}[A(u_k), u_k - u] \le 0 \text{ imply } (u_k) \to u \text{ in } L^p(0,T;V).$$

(See (10.51).)

**Remark 10.11.** In the case of "non-local" operator $A$ one may consider the following modified problem (instead of (10.41)) which is a generalization of the standard Cauchy problem for functional differential equations (delay equations) in one variable:

$$\hat{u}'(t) + \hat{A}(t,\hat{u}_t) = F(t) \text{ for a.a. } t\in[0,T], \tag{10.55}$$

$$\hat{u}(t) = \psi(t) \text{ for a.a. } t\in[-a,0] \tag{10.56}$$

where $\hat{u}_t$ is defined by

$$\hat{u}_t(s) = \hat{u}(t+s), \quad s\in[-a,0], \quad t>0 \tag{10.57}$$

Here $\psi \in L^p(-a,0;V)$, $F\in L^q(0,T;V^\star)$ are given functions and we want to find a function $\hat{u}\in L^p(-a,T;V)$ such that $\hat{u}'\in L^q(0,T;V^\star)$ and $\hat{u}$ satisfies (10.55), (10.56). The operator

$$\hat{A} : (0,T)\times L^p(-a,0;V) \to L^q(0,T;V^\star)$$

is defined by

$$[\hat{A}(t,\hat{u}_t), v] = \tag{10.58}$$

$$\int_{Q_T} \left\{ \sum_{j=1}^n a_j(t,x,\hat{u},D\hat{u};\hat{u}_t)D_j v + a_0(t,x,\hat{u},D\hat{u};\hat{u}_t)v \right\} dtdx,$$

where $v \in L^p(0,T;V)$ and the functions

$$a_j : Q_T \times \mathbb{R}^{n+1} \times L^p(-a,0;V) \to \mathbb{R}$$

satisfy conditions which are analogous to $(C1) - (C5)$, with $L^p(-a,0;V)$ instead of $L^p(0,T;V)$ and $L^p(-a,0;W^{1-\delta,p}(\Omega))$ instead of $L^p(0,T;W^{1-\delta,p}(\Omega))$.

Problem (10.55), (10.56) can be reduced to problem of the form (10.41), in the case when $\psi \in L^p(-a,0;V)$ satisfies $\psi' \in L^q(-a,0;V^\star)$. Indeed, assume that $\hat{u} \in L^p(-a,T;V)$ satisfies (10.55), (10.56) such that $\hat{u}' \in L^q(-a,T;V^\star)$ and define $u$ and $\tilde{u}$ by

$$u(t) = \hat{u}(t) \text{ for } t \in (0,T), \tag{10.59}$$

$$\tilde{u}(t) = u(t) \text{ for } t \in (0,T) \text{ and } \tilde{u}(t) = \psi(t) \text{ for } t \in (-a,0). \tag{10.60}$$

Further, define operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ by

$$[A(u)](t) = \hat{A}(t,\tilde{u}_t), \quad u \in L^p(0,T;V) \tag{10.61}$$

where $\tilde{u}$ is defined by (10.60). Since for $\hat{u} \in L^p(-a,T;V)$ we have $\hat{u}' \in L^q(-a,T;V^\star)$, function $u \in L^p(0,T;V)$, defined by (10.59) satisfies

$$u'(t) + [A(u)](t) = F(t), \quad t \in (0,T) \tag{10.62}$$

$$u(0) = \psi(0). \tag{10.63}$$

Conversely, if $u \in L^p(0,T;V)$ satisfies (10.62), (10.63) then $\hat{u}$, defined by

$$\hat{u}(t) = u(t), \quad t \in (0,T), \quad \hat{u}(t) = \psi(t), \quad t \in (-a,0)$$

satisfies (10.55), (10.56) and $\hat{u} \in L^p(-a,T;V)$, $\hat{u}' \in L^q(-a,T;V^\star)$.

Further, if the functions $a_j$ in (10.58) satisfy the above mentioned conditions (which are analogous to $(C1) - (C5)$), then the functions defining operator $A$ by (10.61), satisfy $(C1) - (C5)$). Consequently, by Theorem 10.9 we obtain existence of solutions of (10.62), (10.63) (since $\psi(0) \in L^2(\Omega)$) which implies the existence of solutions to (10.55), (10.56).

**Example 10.12.** Now we formulate examples satisfying $(C1) - (C5)$), i.e. assumptions of Theorem 10.9. Let $a_i$ have the form

$$a_j(t,x,\eta,\zeta;u) = b(t,x,[H(u)](t,x))\xi_j|\zeta|^{p-2}, \quad j = 1,\ldots,n,$$

$$a_0(t,x,\eta,\zeta;u) = b_0(t,x,[H_0(u)](t,x))\eta|\eta|^{p-2} + \hat{b}_0(t,x,[F_0(u)](t,x))\hat{\alpha}_0(t,x,\eta,\zeta)$$

where $b, b_0, \hat{b}_0, \hat{\alpha}_0$ are Carathéodory functions and they satisfy

$$\text{const} \geq b(t,x,\theta) \geq \frac{c_2}{1+|\theta|^{\sigma^\star}}, \quad \text{const} \geq b_0(t,x,\theta) \geq \frac{c_2}{1+|\theta|^{\sigma^\star}}$$

with some positive constants $c_2$ and $0 \le \sigma^\star < p-1$,

$$|\hat{b}_0(t,x,\theta)| \le 1 + |\theta|^{p-1-\varrho^\star} \text{ with } 0 < \varrho^\star < p-1 \text{ and}$$

$$|\hat{\alpha}_0(t,x,\eta,\zeta)| \le c_1(1 + |\eta|^{\hat{\varrho}} + |\zeta|^{\hat{\rho}})$$

with some constants $c_1$, $\hat{\varrho} \ge 0$, $\sigma^\star + \hat{\varrho} < \varrho^\star$.

Finally,

$$H, H_0 : L^p(0,T;W^{1-\delta,p}(\Omega)) \to C(\overline{Q_T}), \quad F_0 : L^p(0,T;W^{1-\delta,p}(\Omega)) \to L^p(Q_T),$$

are linear and continuous operators. Thus, $[H(u)](t,x)$ and $[H_0(u)](t,x)$ may have one of the forms

$$\int_{Q_t} d(t,x,\tau,\xi)u(\tau,\xi)d\tau d\xi \text{ where } d \text{ is continuous in } (t,x),$$

$$\sup_{(t,x)\in Q_T} \int_{Q_T} |d(t,x,\tau,\xi)|^q d\tau d\xi < \infty,$$

$$\int_{\Gamma_t} d(t,x,\tau,\xi)u(\tau,\xi)d\tau d\sigma_\xi \text{ where } d \text{ is continuous in } (t,x),$$

$$\sup_{(t,x)\in Q_T} \int_{\Gamma_T} |d(t,x,\tau,\xi)|^q d\tau d\sigma_\xi < \infty, \quad \Gamma_t = [0,t) \times \partial\Omega.$$

To prove that examples of the above type satisfy the conditions $(C1)-(C5))$, we apply similar arguments as in Example 4.19.

Clearly, assumptions $(C1)$, $(C3)$, $(C5)$ hold. In order to show $(C2)$, we only have to show that the second term in $a_0(t,x,\eta,\zeta;u)$ satisfies the desired inequality. By Young's inequality we obtain

$$|\hat{b}_0(t,x,[F_0(u)](t,x))\hat{\alpha}_0(t,x,\eta,\zeta)| \le [1 + |F_0(u)|^{p-1-\varrho^\star}]c_1(1 + |\eta|^{\hat{\varrho}} + |\zeta|^{\hat{\varrho}}) \le$$

$$\text{const}(1 + |\eta|^{\hat{\varrho}} + |\zeta|^{\hat{\varrho}})^{p_1} + \text{const}[1 + |F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

where

$$p_1 = \frac{p-1}{\hat{\varrho}} > 1 \text{ and } q_1 = \frac{p_1}{p_1 - 1} = \frac{p-1}{p-1-\hat{\varrho}}.$$

Consequently, we obtain for this term $(C2)$ with

$$k_1(u) = \text{const}[1 + |F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

since by Hölder's inequality we have for this term

$$\int_{Q_T} |k_1(u)|^q dtdx = \text{const} \int_{Q_T} [1 + |F_0(u)|^{(p-1-\varrho^\star)q_1 q}]dtdx \le$$

$$\text{const}\left[1 + \int_{Q_T} |F_0(u)|^p dtdx\right]^{\mu/p} \le \text{const}\left[1 + \|u\|^{\mu}_{L^p(0,T;V)}\right]$$

where

$$\mu = (p-1-\varrho^\star)q_1 q = \frac{p-1-\varrho^\star}{p-1-\hat{\varrho}} p < p.$$

Now we prove that $(C4)$ is satisfied. Clearly, for our example we have in (10.38)

$$[g_2(u)](t) = \min\left\{ \frac{\text{const}}{1+\|H(u)\|^{\sigma^\star}_{C(\overline{Q_T})}}, \frac{\text{const}}{1+\|H_0(u)\|^{\sigma^\star}_{C(\overline{Q_T})}} \right\} \geq$$

$$\text{const}\left[1+\|u\|_{L^p(0,T;V)}\right]^{-\sigma^\star}.$$

Further, by Young's inequality

$$|\hat{b}_0(t,x,[F_0(u)](t,x))\hat{\alpha}_0(t,x,\eta,\zeta)\eta| \leq$$

$$[1+|F_0(u)|^{p-1-\varrho^\star}]\text{const}(1+|\eta|^{\hat{\varrho}+1}+|\zeta|^{\hat{\varrho}+1}) \leq$$

$$\frac{\varepsilon^p}{p}(1+|\eta|^{p-\sigma^\star}+|\zeta|^{p-\sigma^\star}) + C(\varepsilon)[1+|F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

for any $\varepsilon > 0$ (because $\hat{\varrho}+1 < p-\sigma^\star$) where

$$q_1 = \frac{p_1}{p_1-1} = \frac{p-\sigma^\star}{p-\sigma^\star-\hat{\varrho}-1}, \quad p_1 = \frac{p-\sigma^\star}{\hat{\varrho}+1},$$

and $C(\varepsilon)$ is a constant depending on $\varepsilon$. Choosing sufficiently small $\varepsilon > 0$, we obtain $(C4)$ with

$$[k_2(u)](t,x) = C(\varepsilon)[1+|F_0(u)|^{(p-1-\varrho^\star)q_1}]$$

since

$$\|k_2(u)\|_{L^1(Q_t)} = \text{const}\int_{Q_t}[1+|F_0(u)|^{(p-1-\varrho^\star)q_1}] =$$

$$\text{const}\int_{Q_t}[1+|F_0(u)|^{(p-\sigma^\star)\lambda}] \leq$$

$$\text{const}\left[1+\|F_0(u)\|^{(p-\sigma^\star)\lambda}_{L^p(Q_t)}\right] \leq \text{const}\left[1+\|u\|^{(p-\sigma^\star)\lambda}_{L^p(0,t;W^{1-\delta,p}(\Omega))}\right] \leq$$

$$\text{const}\left[1+\|u\|_{L^p(0,t;V)}\right]^{\sigma} \text{ with } \sigma = (p-\sigma^\star)\lambda \text{ where}$$

$$\lambda = \frac{p-1-\varrho^\star}{p-1-\sigma^\star-\hat{\varrho}} < 1, \text{ because}$$

$$\sigma^\star + \hat{\varrho} < \varrho^\star \text{ and thus } (p-\sigma^\star)\lambda < p-\sigma^\star.$$

If the functions $b, b_0$ are between two positive constants, then, it is not difficult to show that $(C1)$ – $(C5)$ are fulfilled when

$$H, H_0 : L^p(0,T;W^{1-\delta,p}(\Omega)) \to L^p(Q_T)$$

are continuous linear operators (like $F_0$). So in this case $[H(u)](t,x)$, $[H_0(u)](t,x)$ (and also $[F_0(u)](t,x)$) may have also the forms

$$\int_0^t d(t,x,\tau)u(\tau,x)d\tau, \quad \int_\Omega d(t,x,\xi)u(t,\xi)d\xi$$

where

$$\int_0^T \sup_{x\in\Omega}\left[\int_0^T |d(t,x,\tau)|^q d\tau\right]^{p/q} dt < \infty, \quad \int_\Omega \sup_{t\in[0,T]}\left[\int_\Omega |d(t,x,\xi)|^q d\xi\right]^{p/q} dx < \infty,$$

respectively, or

$$u(\chi(t),x) \text{ where } \chi \in C^1[0,T], \quad \chi' > 0, \quad 0 \le \chi(t) \le t.$$

## Problems

1. Prove Remark 10.5.

2. Prove Remark 10.10.

3. Show that if the functions $a_j$ satisfy $(B1)$, $(B2)$, $(\tilde{B}3)$, $(B4)$ and there is $w_0 \in W^{1,p}(\Omega)$ such that $w_0|_{\partial\Omega} = g$ then there is a weak solution $u = \tilde{u}+u_0$ of the initial-boundary value problem with nonhomogeneous boundary condition, formulated in Problem **4** of Section 8 (where $u_0(t,x) = w_0(x)$).

4. Let $V = W_0^{m,p}(\Omega)$ where $m \ge 1, p \ge 2$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary. Consider the operator $A$ defined by

$$[A(u),v] = \int_0^T \langle [\tilde{A}(t)][u(t)], v(t)\rangle dt =$$

$$\int_0^T \left\{\int_\Omega \left[\sum_{|\alpha|=m} (D^\alpha u)|D^\alpha u|^{p-2} D^\alpha v\right] dx\right\} dt +$$

$$\int_0^T \left\{\int_\Omega \left[\sum_{|\alpha|<m} a_\alpha(t,x,\dots,D^\beta u,\dots) D^\alpha v\right] dx\right\} dt, \quad |\beta| \le m,$$

$u, v \in L^p(0,T;V)$ where the functions $a_\alpha$ ($|\alpha| < m$) satisfy the Carathéodory conditions and there exist a constant $c_1 > 0$ and $k_1 \in L^q(\Omega)$ such that

$$|a_\alpha(t,x,\xi)| \le c_1|\xi|^{p-1} + k_1(x) \text{ for } \xi \in \mathbb{R}^N, \text{ a.a. } x \in \Omega.$$

Further,

$$a_\alpha(t,x,\xi)\xi_\alpha \ge 0 \text{ for } \xi \in \mathbb{R}^N, \text{ a.a. } x \in \Omega.$$

(See the notations in Problem 5 in Section 4.)

By using the arguments of the proof of Theorem 10.4, show that for arbitrary $F \in L^q(0,T;V^\star)$, $u_0 \in L^2(\Omega)$ there exists a solution $u$ of problem (7.1).

# 11 Existence of solutions for $t \in (0, \infty)$

In this section we shall prove existence of solutions to nonlinear evolution equations in infinite time horizon. These results will be applied to nonlinear parabolic differential equations and functional parabolic equations which were considered in Sections 8 and 10.

First we formulate some basic definitions.

**Definition 11.1.** *Let $V$ be a Banach space, $1 \le p < \infty$. The set $L^p_{loc}(0, \infty; V)$ consists of all functions $f : (0, \infty) \to V$ for which the restriction $f|_{(0,T)}$ of $f$ to $(0, T)$ belongs to $L^p(0, T; V)$ for each finite $T > 0$.*

*Further, by using the notations $Q_\infty = (0, \infty) \times \Omega$, $\Gamma_\infty = (0, \infty) \times \partial\Omega$, denote by $L^p_{loc}(Q_\infty)$ and $L^p_{loc}(\Gamma_\infty)$ the set of functions $f : Q_\infty \to \mathbb{R}$ and $g : \Gamma_\infty \to \mathbb{R}$, respectively, for which $f|_{Q_T} \in L^p(Q_T)$, $g|_{\Gamma_T} \in L^p(\Gamma_T)$ for arbitrary finite $T > 0$.*

First we consider the case when $A : L^p_{loc}(0, \infty; V) \to L^q_{loc}(0, \infty; V^\star)$ is "local", i.e. it has the form $[A(u)](t) = [\tilde{A}(t)][u(t)]$ where for fixed $t$, $\tilde{A}(t)$ maps $V$ into $V^\star$.

**Theorem 11.2.** *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$. Assume that for almost all $t > 0$, $\tilde{A}(t) : V \to V^\star$ is such that operator $A : L^p_{loc}(0, \infty; V) \to L^q_{loc}(0, \infty; V^\star)$, defined by $[A(u)](t) = [\tilde{A}(t)][u(t)]$ satisfies the assumptions of Theorems 7.1, 9.2, respectively for each fixed $T > 0$.*

*Then for any $F \in L^q_{loc}(0, \infty; V^\star)$ and $u_0 \in H$ there exists $u \in L^p_{loc}(0, \infty; V)$ such that $u' \in L^q_{loc}(0, \infty; V^\star)$,*

$$u'(t) + [A(u)](t) = F(t) \text{ for a.a. } t \in (0, \infty), \quad u(0) = u_0. \tag{11.1}$$

*In the case when the conditions of Theorem 7.1 are fulfilled (monotone case), the solution of (11.1) is unique.*

*Proof.* Let $(T_j)$ be an increasing sequence of positive numbers with $\lim(T_j) = +\infty$. Due to Theorems 7.1, 9.2, respectively, there exist $u_j \in L^p(0, T_j; V)$ such that $u'_j \in L^q(0, T_j; V^\star)$ and

$$u'_j(t) + [A(u_j)](t) = F(t) \text{ for a.a. } t \in [0, T_j], \quad u_j(0) = u_0. \tag{11.2}$$

The coercivity assumptions (7.3), (9.6), respectively, imply that for all fixed $T > 0$ (and sufficiently large $j$) $u_j|_{[0,T_j]}$ is bounded in $L^p(0, T; V)$. The (boundedness) assumptions (7.2), (9.5) imply that $[A(u_j)]|_{[0,T_j]}$ is bounded in $L^q(0, T; V^\star)$ for all fixed finite $T > 0$.

Therefore, by using a "diagonal process", one can select a subsequence of $(u_j)$ (again denoted by $(u_j)$, for simplicity) such that for each fixed $k$, $u_j|_{[0,T_k]}$ is weakly convergent in $L^p(0, T_k; V)$ and the sequence $u'_j|_{[0,T_k]}$ is weakly convergent in $L^q(0, T_k; V^\star)$ as $j \to \infty$. Thus we obtain a function $u \in L^p_{loc}(0, \infty; V)$ such that $u' \in L^q_{loc}(0, \infty; V^\star)$, $u(0) = u_0$, further, for each fixed $k$

$$(u_j|_{[0,T_k]}) \to u|_{[0,T_k]} \text{ weakly in } L^p(0, T_k; V), \tag{11.3}$$

$$(u_j'|_{[0,T_k]}) \to u'|_{[0,T_k]} \text{ weakly in } L^q(0, T_k; V^\star). \tag{11.4}$$

Thus, similarly to the proof of (7.16) (see Lemma 7.3), one obtains $u(0) = u_0$ (by using $u_j(0) = u_0$). Further, by (11.2) for $j \geq k$

$$u_j'(t) + [A(u_j)](t) = F(t) \text{ for a.a. } t \in [0, T_k], \quad u_j(0) = u_0, \tag{11.5}$$

thus by Remark 6.7 and (11.4)

$$\int_0^{T_k} \langle [A(u_j)](t), u_j(t) - u(t) \rangle dt =$$

$$\int_0^{T_k} \langle F(t), u_j(t) - u(t) \rangle dt - \int_0^{T_k} \langle u_j'(t), u_j(t) - u(t) \rangle dt =$$

$$\int_0^{T_k} \langle F(t), u_j(t) - u(t) \rangle dt - \frac{1}{2} \| u_j(T_k) - u(T_k) \|_H^2 +$$

$$\int_0^{T_k} \langle u'(t), u_j(t) - u(t) \rangle dt,$$

hence

$$\limsup_{j \to \infty} \int_0^{T_k} \langle [A(u_j)](t), u_j(t) - u(t) \rangle dt \leq 0.$$

Since for fixed $k$ $A$ is pseudomonotone with respect to $W_p^1(0, T_k; V, H)$ (as operator from $L^p(0, T_k; V)$ into $L^q(0, T_k; V^\star)$),

$$(A(u_j)) \to A(u) \text{ weakly in } L^q(0, T_k; V^\star)$$

and so from (11.4), (11.5) we obtain as $j \to \infty$ that (11.1) holds for a.a. $t \in [0, T_k]$. Since it holds for all $k$ and $\lim_{k \to \infty} T_k = +\infty$, we obtain (11.1) for a.a. $t \in (0, \infty)$.

In the case when the conditions of Theorem 7.1 are fulfilled (monotone case), $u_j$ is unique for all $j$ and thus the solution $u$ of (11.1) is unique, too. (The restriction of a solution in $(0, \infty)$ to $(0, T_j)$ satisfies the initial value problem in $(0, T_j)$.) □

Now we consider the case when operator $A$ is "non-local", i.e. $[A(u)](t)$ depends not only on $u(t)$. Then it is important to assume that $A$ has the "Volterra property".

**Definition 11.3.** *An operator $A : L^p_{loc}(0, \infty; V) \to L^q_{loc}(0, \infty; V^\star)$ is of* Volterra type *(it has the* Volterra property*) if for each $u \in L^p_{loc}(0, \infty; V)$ and $t > 0$, $[A(u)](t)$ depends only on $u|_{(0,t)}$, i.e. the restriction of $u$ to $(0, t)$.*

*If $A : L^p_{loc}(0, \infty; V) \to L^q_{loc}(0, \infty; V^\star)$ is of Volterra type, then the "restriction of $A$ to $[0, T]$", denoted by $A_T$, is the operator $A_T : L^p(0, T; V) \to L^q(0, T; V^\star)$, defined by*

$$A_T(u) = A(u_T), \quad u \in L^p(0, T; V) \text{ where}$$

$$u_T(t) = u(t) \text{ for } t \in [0, T] \text{ and } u_T(t) = 0 \text{ for } t > T.$$

**Theorem 11.4.** *Let the operator* $A : L^p_{loc}(0,\infty;V) \to L^q_{loc}(0,\infty;V^\star)$ *be an operator of Volterra type such that for each finite* $T > 0$, *the restriction of* $A$ *to* $[0,T]$, $A_T : L^p(0,T;V) \to L^q(0,T;V^\star)$ *satisfies the assumptions of Theorem 9.6, i.e. it is bounded, demicontinuous, pseudomonotone with respect to* $W^1_p(0,T;V,H)$ *and it is coercive in the sense of Theorem 9.6.*

*Then for arbitrary* $F \in L^q_{loc}(0,\infty;V^\star)$, $u_0 \in H$ *there exists* $u \in L^p_{loc}(0,\infty;V)$ *such that* $u' \in L^q_{loc}(0,\infty;V^\star)$ *and*

$$u'(t) + [A(u)](t) = F(t) \text{ for a.a. } t \in (0,\infty), \quad u(0) = u_0. \tag{11.6}$$

*Proof.* Similarly to the proof of Theorem 11.2, let $(T_j)$ be an increasing sequence of positive numbers with $\lim(T_j) = +\infty$. Due to Theorem 9.6 there exist functions $u_j \in L^p(0,T_j;V)$ such that $u_j' \in L^q(0,T_j;V^\star)$ and

$$u_j'(t) + [A_{T_j}(u_j)](t) = F(t) \text{ for a.e. } t \in [0,T_j], \quad u_j(0) = u_0.$$

The Volterra property implies that $v = u_j|_{[0,T_k]}$ satisfies

$$v'(t) + [A_{T_k}(v)](t) = F(t) \text{ for a.a. } t \in [0,T_k] \tag{11.7}$$

if $T_k < T_j$. Coercivity of $A_T$ implies that for all fixed finite $T > 0$ (and sufficiently large $j$), $u_j|_{[0,T]}$ is bounded in $L^p(0,T;V)$. From the boundedness of $A_T$ it follows that $A_T\left(u_j|_{[0,T]}\right)$ is bounded in $L^q(0,T;V^\star)$.

Therefore, by a "diagonal process", one can select a subsequence of $(u_j)$ (again denoted by $(u_j)$) such that for each fixed $k$,

$$\left(u_j|_{[0,T_k]}\right) \text{ is weakly convergent in } L^p(0,T_k;V) \text{ and}$$

$$\left(u_j'|_{[0,T_k]}\right) \text{ is weakly convergent in } L^q(0,T_k;V^\star) \text{ as } j \to \infty.$$

Thus we obtain a function $u \in L^p_{loc}(0,\infty;V)$ such that $u' \in L^q_{loc}(0,\infty;V^\star)$. By using the arguments of the proof of (7.16) (see Lemma 7.3), we obtain $u(0) = u_0$. Further,

$$\left(u_j|_{[0,T_k]}\right) \to u|_{[0,T_k]} \text{ weakly in } L^p(0,T_k;V) \text{ and} \tag{11.8}$$

$$\left(u_j'|_{[0,T_k]}\right) \to u'|_{[0,T_k]} \text{ weakly in } L^q(0,T_k;V^\star) \text{ as } j \to \infty. \tag{11.9}$$

Since by (11.7)

$$u_j'(t) + [A_{T_k}(u_j)](t) = F(t) \text{ for a.a. } t \in [0,T_k], \quad u_j(0) = u_0, \tag{11.10}$$

by Remark 6.7

$$\int_0^{T_k} \langle [A_{T_k}(u_j)](t), u_j(t) - u(t)\rangle dt =$$

$$\int_0^{T_k} \langle F(t), u_j(t) - u(t)\rangle dt - \int_0^{T_k} \langle u_j'(t), u_j(t) - u(t)\rangle dt =$$

$$\int_0^{T_k} \langle F(t), u_j(t) - u(t)\rangle dt -$$

$$\frac{1}{2}\|u_j(T_k) - u(T_k)\|_H^2 + \int_0^{T_k} \langle u'(t), u_j(t) - u(t)\rangle dt,$$

hence

$$\limsup_{j\to\infty} \int_0^{T_k} \langle [A_{T_k}(u_j)](t), u_j(t) - u(t)\rangle dt \le 0. \tag{11.11}$$

Since (for fixed $k$) $A_{T_k} : L^p(0, T_k; V) \to L^q(0, T_k; V^\star)$ is pseudomonotone with respect to $W_p^1(0, T; V, H)$, the inequality (11.11), (11.8), (11.9) imply that

$$(A_{T_k}(u_j)) \to A_{T_k}(u) \text{ weakly in } L^q(0, T_k; V^\star) \text{ as } j \to \infty.$$

Thus from (11.10) we obtain as $j \to \infty$

$$u'(t) + [A_{T_k}(u)](t) = F(t) \text{ for a.a. } t \in [0, T_k], \quad u(0) = u_0, \tag{11.12}$$

(11.12) holds for all $k$, so we have (11.6). □

Now we apply Theorem 11.2 to operators of the form (8.1) where $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, $2 \le p < \infty$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with "sufficiently good" boundary, $H = L^2(\Omega)$.

Assume that

($B_\infty 1$) Functions $a_j : Q_\infty \times \mathbb{R}^{n+1} \to \mathbb{R}$ ($j = 1, ..., n$) satisfy the Carathéodory conditions.

($B_\infty 2$) There exist a constant $c_1$ and a function $k_1 \in L^q_{loc}(Q_\infty)$ ($1/p + 1/q = 1$) such that for a.a. $(t, x) \in Q_\infty$, all $\xi \in \mathbb{R}^{n+1}$

$$|a_j(t, x, \xi)| \le c_1|\xi|^{p-1} + k_1(t, x).$$

($B_\infty 3$) For a.a. $(t, x) \in Q_\infty$, all $\xi, \xi^\star \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n} [a_j(t, x, \xi) - a_j(t, x, \xi^\star)](\xi_j - \xi_j^\star) \ge 0.$$

($B_\infty 4$) There exist a constant $c_2 > 0$, $k_2 \in L^1_{loc}(Q_\infty)$ such that for a.e. $(t, x) \in Q_\infty$, all $\xi \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n} a_j(t, x, \xi)\xi_j \ge c_2|\xi|^p - k_2(t, x).$$

From Theorems 7.1, 11.2 directly follows

**Theorem 11.5.** *Assume ($B_\infty 1$) – ($B_\infty 4$). Then for all $F \in L^q_{loc}(0, \infty; V^\star)$, $u_0 \in L^2(\Omega)$ there is a unique $u \in L^p_{loc}(0, \infty; V)$ such that $u' \in L^q_{loc}(0, \infty; V^\star)$ and*

$$u'(t) + [A(u)](t) = F(t) \textit{ for a.a. } t \in (0, \infty), \quad u(0) = u_0 \tag{11.13}$$

*with the operator (8.1).*

Instead of $(B_\infty 3)$ assume

$(\tilde{B}_\infty 3)$ There exists a constant $\tilde{c} > 0$ such that for a.a. $(t,x) \in Q_\infty$, all $\eta \in \mathbb{R}$, $\zeta, \zeta^\star \in \mathbb{R}^n$

$$\sum_{j=1}^n [a_j(t,x,\eta,\zeta) - a_j(t,x,\eta,\zeta^\star)](\xi_j - \xi_j^\star) \geq \tilde{c}|\zeta - \zeta^\star|^p.$$

From Theorems 10.4, 11.2 one obtains

**Theorem 11.6.** *Assume $(B_\infty 1)$, $(B_\infty 2)$, $(\tilde{B}_\infty 3)$ $(B_\infty 4)$. Then for all $F \in L^q_{loc}(0,\infty;V^\star)$, $u_0 \in L^2(\Omega)$ there is a solution $u \in L^p_{loc}(0,\infty;V)$ of (11.13) with the operator (8.1).*

If instead of $(B_\infty 3)$ we assume

$(B_\infty 3')$ For a.a. $(t,x) \in Q_\infty$, all $\eta \in \mathbb{R}$, $\zeta, \zeta^\star \in \mathbb{R}^n$, $\zeta \neq \zeta^\star$

$$\sum_{j=1}^n [a_j(t,x,\eta,\zeta) - a_j(t,x,\eta,\zeta^\star)](\xi_j - \xi_j^\star) > 0$$

we obtain from Theorems 10.6, 11.2

**Theorem 11.7.** *Assume $(B_\infty 1)$, $(B_\infty 2)$, $(B_\infty 3')$ $(B_\infty 4)$. Then for all $F \in L^q_{loc}(0,\infty;V^\star)$, $u_0 \in L^2(\Omega)$ there is a solution $u \in L^p_{loc}(0,\infty;V)$ of (11.13) with the operator (8.1).*

**Remark 11.8.** If $V = W_0^{1,p}(\Omega)$ and $\Omega$ is bounded then instead of $(B_\infty 4)$ it is sufficient to assume

$(B_\infty 4')$ For a.e. $(t,x) \in Q_\infty$, all $\xi = (\eta,\zeta) \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^n a_j(t,x,\xi)\xi_j \geq c_2|\zeta|^p - k_2(t,x)$$

with some constant $c_2 > 0$, $k_2 \in L^1_{loc}(Q_\infty)$. (See Remarks 3.9, 4.11.)

Now we apply Theorem 11.4 to operators of the form (10.37) (see Definition 10.8) where $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, $2 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ is a bounded domain (with sufficiently smooth boundary).

**Theorem 11.9.** *Assume that the functions*

$$a_j : Q_\infty \times \mathbb{R}^{n+1} \times L^p_{loc}(0,\infty;V) \to \mathbb{R}, \quad j = 0,1,...,n$$

*have the Volterra property, i.e. for all $t > 0$, $a_j(t,x,\eta,\zeta;u)$ depends only on the restriction $u\,|_{(0,t)}$ of $u$ to $(0,t)$. Further, for all finite $T > 0$, the restrictions of $a_j$ to $Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V)$ satisfy (C1) – (C5), i.e. assumptions of Theorem 10.9.*

*Then for arbitrary $F \in L^q_{loc}(0,\infty;V^\star)$, $u_0 \in L^2(\Omega)$ there exists a function $u \in L^p_{loc}(0,\infty;V)$ such that $u' \in L^q_{loc}(0,\infty;V^\star)$ and (11.6) holds with the operator $A$ of the form (10.37) with $h = 0$, i.e. when $A$ is defined by*

$$\langle [A(u)](t), w\rangle = \tag{11.14}$$

$$\int_{\Omega} \left[ \sum_{j=1}^{n} a_j(t,x,u,Du;u) D_j w + a_0(t,x,u,Du;u) w \right] dx,$$

*where* $u \in L^p_{loc}(0,\infty;V)$, $v \in V$.

## Problems

1. Prove Theorem 11.5.
2. Prove Theorem 11.6.
3. Prove Remark 11.8.
4. Consider the functions
$$\alpha_j : Q_\infty \times \mathbb{R} \times \mathbb{R}, \quad j = 0,1,\ldots,n$$
which satisfy the conditions of Problem 1 in Section 8 for all $t \in (0,\infty)$, with the same constants. Prove that there exists a unique solution of problem (11.1) with the operator $\tilde{A}(t)$ defined by functions $\alpha_j$ in Problem 1 of Section 8.
5. Formulate and prove an existence and uniqueness theorem for the solution of (11.1) where the operator $\tilde{A}(t)$ is defined in Problem 3 of Section 8.
6. Formulate and prove an existence and uniqueness theorem for the solution of (11.1) where the operator $\tilde{A}(t)$ is defined in Problem 6 of Section 8.
7. Formulate and prove an existence theorem for the solution of (11.1) where the operator $\tilde{A}(t)$ is defined in Problem 4 of Section 10.

# 12 Qualitative properties of the solutions

## Boundedness of solutions

First we formulate and prove theorems on the boundedness of $\|u(t)\|_H$, $t \in (0,\infty)$ for the solutions $u$ of (11.1) and (11.6).

**Theorem 12.1.** *Assume that the operator* $A : L^p_{loc}(0,\infty;V) \to L^q_{loc}(0,\infty;V^\star)$ *is given by*

$$[A(u)](t) = [\tilde{A}(t)][u(t)] \text{ with } \tilde{A}(t) : V \to V^\star$$

*and the assumptions of Theorem 11.2 are fulfilled such that for a.a.* $t \in (0,\infty)$, $v \in V$

$$\langle [\tilde{A}(t)](v), v \rangle \geq c_2 \|v\|_V^p - k_2(t) \text{ where } k_2 \in L^\infty(0,\infty) \tag{12.1}$$

*(i.e. the function* $k_2$ *in (9.6) is essentially bounded) and* $\|F(t)\|_{V^\star}$ *is bounded for a.e.* $t \in (0,\infty)$, *i.e.* $F \in L^\infty(0,\infty;V^\star)$.

*Then for a solution $u$ of (11.1), $\|u(t)\|_H$ is bounded for $t \in (0, \infty)$, so $u \in L^\infty(0, \infty; H)$ and*

$$\int_{T_1}^{T_2} \|u(t)\|_V^p dt \le c_3(T_2 - T_1) \text{ for } 0 < T_1 < T_2 \tag{12.2}$$

*with some constant $c_3$ (not depending on $T_1, T_2$).*

*Proof.* Let $u$ be a solution of (11.1) and $y(t) = \|u(t)\|_H^2$. Then by (11.1), (12.1) and Young's inequality for arbitrary $\varepsilon > 0$

$$\langle u'(t), u(t)\rangle + \langle [\tilde{A}(t)][u(t)], u(t)\rangle = \langle F(t), u(t)\rangle,$$

hence

$$\langle u'(t), u(t)\rangle + c_2\|u(t)\|_V^p - k_2(t) \le \|F(t)\|_{V^\star}\|u(t)\|_V \le \tag{12.3}$$
$$\varepsilon\|u(t)\|_V^p + C(\varepsilon)\|F(t)\|_{V^\star}^q.$$

Since by Remark 6.7

$$\int_{T_1}^{T_2} \langle u'(t), u(t)\rangle dt = \frac{1}{2}[\|u(T_2)\|_H^2 - \|u(T_1)\|_H^2],$$

choosing sufficiently small $\varepsilon > 0$ and integrating (12.3) with respect to $t$ over $[T_1, T_2]$, we obtain

$$\frac{1}{2}[\|u(T_2)\|_H^2 - \|u(T_1)\|_H^2] + \frac{c_2}{2}\int_{T_1}^{T_2} \|u(t)\|_V^p dt \le \tag{12.4}$$

$$\int_{T_1}^{T_2} k_2(t)dt + c_4\int_{T_1}^{T_2} \|F(t)\|_{V^\star}^q dt \le c_5(T_2 - T_1).$$

Since the imbedding $V \subset H$ is continuous,

$$y(t) = \|u(t)\|_H^2 \le \text{const}\|u(t)\|_V^2,$$

thus (12.4) implies

$$y(T_2) - y(T_1) + c^\star \int_{T_1}^{T_2} [y(t)]^{p/2} dt \le 2c_5(T_2 - T_1) \tag{12.5}$$

with some positive constant $c^\star$.

We show that the inequality (12.5) implies that $y(t)$ is bounded for $t \in (0, \infty)$. Indeed, assuming that the (continuous) function $y$ is not bounded, for any $M > 0$ there are $t_0 > 0$ and $t_1 \in [0, t_0]$ such that

$$y(t_1) = \max_{[0,t_0]} y > M.$$

Since $y$ is continuous, there is $\delta > 0$ such that

$$y(t) > M \text{ if } t_1 - \delta \le t < t_1,$$

hence by (12.5)

$$y(t_1) - y(t_1 - \delta) + c^\star \delta M^{p/2} \le 2c_5\delta$$

which is impossible for all $M > 0$, because $y(t_1) - y(t_1 - \delta) \ge 0$ and $p > 1$. Finally, from (12.4) and the boundedness of $y(t)$ we obtain (12.2). $\square$

**Theorem 12.2.** *Assume that the conditions of Theorem 11.4 are fulfilled such that for a.a. $t \in (0,\infty)$, $v \in L^p_{loc}(0,\infty;V)$ with $v' \in L^q_{loc}(0,\infty;V^\star)$*

$$\langle [\tilde{A}(t)][v(t)], v(t)\rangle \ge c_2\|v(t)\|_V^p - c_3\left[\sup_{\tau\in[0,t]}\|v(\tau)\|_H^{p_1} + \varphi(t)\sup_{\tau\in[0,t]}\|v(\tau)\|_H^p + 1\right]$$

*holds where $c_2, c_3 > 0$, $0 < p_1 < p$ are constants, $\varphi \ge 0$ is a function with the property $\lim_\infty \varphi = 0$. Further, $\|F(t)\|_{V^\star}$ is bounded for a.a. $t \in (0,\infty)$.*

*Then for a solution $u$ of (11.6) (with arbitrary initial condition), $\|u(t)\|_H$ is bounded for $t \in (0,\infty)$ and (12.2) holds.*

*Proof.* Similarly to the proof of Theorem 12.1, we have for a solution of (11.6)

$$\langle u'(t), u(t)\rangle + c_2\|u(t)\|_V^p - c_3\left[\sup_{\tau\in[0,t]}\|u(\tau)\|_H^{p_1} + \varphi(t)\sup_{\tau\in[0,t]}\|u(\tau)\|_H^p + 1\right] \le$$

$$\varepsilon\|u(t)\|_V^p + C(\varepsilon)\|F(t)\|_{V^\star}^q.$$

Choosing sufficiently small $\varepsilon > 0$ and integrating over $[T_1, T_2]$, by Remark 6.7 we obtain

$$\frac{1}{2}[\|u(T_2)\|_H^2 - \|u(T_1)\|_H^2] + \frac{c_2}{2}\int_{T_1}^{T_2}\|u(t)\|_V^p dt \le \tag{12.6}$$

$$\tilde{c}_3\int_{T_1}^{T_2}\left[\sup_{[0,t]} y^{p_1/2} + \varphi(t)\sup_{[0,t]} y^{p/2} + 1\right]dt$$

with some constant $\tilde{c}_3 > 0$. Since $y(t) = \|u(t)\|_H^2 \le \text{const}\|u(t)\|_V^2$, we obtain from (12.6)

$$y(T_2) - y(T_1) + c^\star\int_{T_1}^{T_2}[y(t)]^{p/2}dt \le \tag{12.7}$$

$$2\tilde{c}_3\int_{T_1}^{T_2}\left[\sup_{[0,t]} y^{p_1/2} + \varphi(t)\sup_{[0,t]} y^{p/2} + 1\right]dt$$

with some positive constant $c^\star$. We show that (12.7) implies the boundedness of $y$.

Assume that $y(t)$ is not bounded. Then for any $M > 0$ there are $t_0 > 0$ and $t_1 \in [0, t_0]$ such that

$$M + 1 \ge y(t_1) = \sup_{[0,t_0]} y > M.$$

As $y$ is continuous, there is a $\delta > 0$ such that

$$y(t) > M \text{ if } t_1 - \delta \le t \le t_1.$$

Hence by (12.7)

$$y(t_1) - y(t_1 - \delta) + c^\star \delta M^{p/2} \leq$$

$$2\tilde{c}_3 \left[ \delta (M+1)^{p_1/2} + (M+1)^{p/2} \int_{t_1-\delta}^{t_1} \varphi(t) dt + \delta \right]$$

which is impossible for all $M > 0$ because $y(t_1) - y(t_1 - \delta) \geq 0$, $p_1 < p$ and $\lim_\infty \varphi = 0$. From the boundedness of $y(t)$ and (12.6) we obtain (12.2). □

Now consider the case when $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, $2 \leq p < \infty$, $H = L^2(\Omega)$. Similarly to the proof of Theorem 12.2, one proves

**Theorem 12.3.** *Assume that the conditions of Theorem 11.9 are fulfilled such that for all for a.a.* $t \in (0, \infty)$, $v \in L^p_{loc}(0, \infty; V)$ *with* $v' \in L^q_{loc}(0, \infty; V^\star)$ *the inequalities*

$$[g_2(v)](t) \geq const \left[ 1 + \sup_{\tau \in [0,t]} \|v(\tau)\|_{L^2(\Omega)} \right]^{-\sigma^\star}, \tag{12.8}$$

$$\int_\Omega [k_2(v)](t,x) dx \leq \tag{12.9}$$

$$const \left[ 1 + \sup_{\tau \in [0,t]} \|v(\tau)\|^\sigma_{L^2(\Omega)} + \varphi(t) \sup_{\tau \in [0,t]} \|v(\tau)\|^{p-\sigma^\star}_{L^2(\Omega)} \right]$$

*hold with some constants,* $0 \leq \sigma^\star < p - 1$, $1 \leq \sigma < p - \sigma^\star$, $\lim_\infty \varphi = 0$ *and* $\|F(t)\|_{V^\star}$ *is bounded for a.e.* $t \in (0, \infty)$.

*Then for a solution* $u$ *of (11.6) with operator* $A$ *given by*

$$\langle [A(u)](t), w \rangle = \tag{12.10}$$

$$\int_\Omega \left[ \sum_{j=1}^n a_j(t,x,u,Du;u) D_j w + a_0(t,x,u,Du;u) w \right] dx,$$

$$u \in L^p_{loc}(0, \infty; V), \quad w \in V,$$

$\|u(t)\|_H$ *is bounded for* $t \in (0, \infty)$ *and (12.2) holds.*

## Stabilization of the solutions

Now we shall formulate conditions which imply results on the stabilization of solutions $u$ to (11.6) as $t \to \infty$. First consider operators defined by

$$[A(u)](t) = [\tilde{A}(t)][u(t)] \text{ where } \tilde{A}(t) : V \to V^\star \tag{12.11}$$

is defined for all $t > 0$.

**Theorem 12.4.** *Assume that the operator $\tilde{A}(t) : V \to V^\star$ satisfies the conditions of Theorem 7.1 for all $t > 0$ such that for any $v, w \in V$*

$$\langle [\tilde{A}(t)](v) - [\tilde{A}(t)](w), v - w\rangle \geq c_2 \|v - w\|_V^p + c_3 \|v - w\|_H^2 \tag{12.12}$$

*with some constants $c_2 > 0, c_3 \geq 0$. (In this case $\tilde{A}(t)$ is uniformly monotone, see Definition 2.15.) Further, there exist $A_\infty : V \to V^\star$ and $F_\infty \in V^\star$, a continuous function $\Phi$ with the property $\lim_\infty \Phi = 0$ and for all $R > 0$ there is a positive number $c_R$ such that for all $v \in V$ with $\|v\|_V \leq R$, $t > 0$ we have*

$$\|[\tilde{A}(t)](v) - A_\infty(v)\|_{V^\star} \leq c_R \Phi(t) \text{ and } \|F(t) - F_\infty\|_{V^\star} \leq \Phi(t). \tag{12.13}$$

*Then for a solution $u$ of (11.1) with operator $A$ of the form (12.11) we have*

$$\lim_{t\to\infty} \|u(t) - u_\infty\|_H = 0, \quad \lim_{T\to\infty} \int_{T-a}^{T+a} \|u(t) - u_\infty\|_V^p dt = 0 \tag{12.14}$$

*for arbitrary fixed $a > 0$, where $u_\infty \in V$ is the unique solution to*

$$A_\infty(u_\infty) = F_\infty. \tag{12.15}$$

*If*

$$\int_0^\infty \Phi(t)^q dt < \infty \tag{12.16}$$

*is satisfied, too, then we have*

$$\int_0^\infty \|u(t) - u_\infty\|_V^p dt < \infty, \quad \int_0^\infty \|u(t) - u_\infty\|_H^2 dt < \infty. \tag{12.17}$$

*Further, if $c_3 > 0$,*

$$\int_T^\infty \|u(t) - u_\infty\|_H^2 dt \leq \tag{12.18}$$

$$const \left\{ e^{-\gamma T} + \int_0^T \left[ e^{-\gamma(T-t)} \int_t^\infty \Phi(\tau)^q d\tau \right] dt \right\}$$

*holds with some constant $\gamma > 0$.*

*Proof.* By (7.2), (12.12), (12.13) the operator $A_\infty : V \to V^\star$ is bounded, strictly monotone and coercive, too, according to Remark 3.7. Further, it is easy to show that by (12.13) $A_\infty$ is hemicontinuous, because $\tilde{A}(t)$ is hemicontinuous and in (12.13) $\Phi$ is not depending on $v$.

Therefore, Theorem 2.14 implies that (12.15) has a unique solution $u_\infty$ for all $F_\infty \in V^\star$. Further, by Theorem 11.2 there exists a unique solution $u$ of (11.1) in $(0, \infty)$. Then by (12.15) one obtains

$$\langle D_t[u(t) - u_\infty], u(t) - u_\infty\rangle + \langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle = \tag{12.19}$$

$$\langle F(t) - F_\infty(u_\infty), u(t) - u_\infty\rangle.$$

The second term on the left-hand side of (12.19) can be estimated by (12.12) and Young's inequality as follows:

$$\langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle \geq \tag{12.20}$$

$$\langle [A(u)](t) - A(u_\infty), u(t) - u_\infty\rangle - |\langle A(u_\infty) - A_\infty(u_\infty), u(t) - u_\infty\rangle| \geq$$

$$c_2\|u(t) - u_\infty\|_V^p + c_3\|u(t) - u_\infty\|_H^2$$

$$-\frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p - C(\varepsilon)\|[A(u_\infty)](t) - A_\infty(u_\infty)\|_{V^\star}^q.$$

Further, for the right-hand side of (12.19) we have

$$|\langle F(t) - F_\infty(u_\infty), u(t) - u_\infty\rangle| \leq \tag{12.21}$$

$$\frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p + C(\varepsilon)\|F(t) - F_\infty\|_{V^\star}^q.$$

Thus, choosing sufficiently small $\varepsilon > 0$, integrating (12.19) over $[T_1, T_2]$, we obtain by Remark 6.7, (12.13), (12.20), (12.21)

$$\frac{1}{2}\|u(T_2) - u_\infty\|_H^2 - \frac{1}{2}\|u(T_1) - u_\infty\|_H^2 + \tag{12.22}$$

$$\frac{c_2}{2}\int_{T_1}^{T_2}\|u(t) - u_\infty\|_V^p dt + c_3\int_{T_1}^{T_2}\|u(t) - u_\infty\|_H^2 dt \leq$$

$$\text{const}\int_{T_1}^{T_2}\left[\|[A(u_\infty)](t) - A_\infty(u_\infty)\|_{V^\star}^q + \|F(t) - F_\infty\|_{V^\star}^q\right] dt \leq$$

$$\text{const}\int_{T_1}^{T_2}\Phi(t)^q dt.$$

Hence, by using the notation $y(t) = \|u(t) - u_\infty\|_H^2$, we obtain with some $c^\star > 0$

$$y(T_2) - y(T_1) + c^\star\int_{T_1}^{T_2}[y(t)]^{p/2}dt + 2c_3\int_{T_1}^{T_2}y(t)dt \leq c_4\int_{T_1}^{T_2}[\Phi(t)]^q dt. \tag{12.23}$$

Since $\Phi(t)^q$ is bounded and the last term on the left-hand side of (12.23) is nonnegative, we obtain form (12.23), as from (12.5), that $y(t)$ is bounded for $t \in (0, \infty)$.

Further, since $\lim_\infty \Phi = 0$, (12.23) implies that

$$\lim_\infty y = 0. \tag{12.24}$$

First we show that

$$\liminf_\infty y = 0. \tag{12.25}$$

Assuming that (12.25) is not valid, there exist $t_0 > 0$ and $\delta > 0$ such that

$$y(t) \geq \delta \text{ for } t > t_0.$$

Further, since $\lim_\infty \Phi = 0$, for arbitrary $\varepsilon > 0$ there exists $t_1$ such that

$$0 \le [\Phi(t)]^q < \varepsilon \text{ for } t > t_1.$$

Choosing sufficiently large $T_1, T_2$, by using the boundedness of $y$, we obtain from (12.23)

$$c^\star \delta^{p/2}(T_2 - T_1) \le c_4 \varepsilon (T_2 - T_1) + c_5, \text{ i.e. } c^\star \delta^{p/2} \le c_4 \varepsilon + \frac{c_5}{T_2 - T_1}$$

with some constant $c_5$, which is impossible if $\varepsilon$ is sufficiently small and $T_2 - T_1$ is sufficiently large. Thus we have (12.25).

Assume that (12.24) is not true. Then there exist $\varepsilon_0 > 0$ and

$$t_1 < t_1^\star < t_2 < t_2^\star < ... \text{ converging to } +\infty$$

such that

$$\lim_{k\to\infty} y(t_k) = 0, \quad y(t_k) < \varepsilon_0, \quad y(t_k^\star) > \varepsilon_0.$$

Since $y$ is continuous, there is $\tilde{t}_k \in (t_k, t_{k+1})$ with

$$y(\tilde{t}_k) = \sup_{t\in[t_k, t_{k+1}]} y(t) \text{ and } y(\tilde{t}_k) > \varepsilon_0.$$

Applying (12.23) to $T_1 = \tilde{t}_k - \delta_k$, $T_2 = \tilde{t}_k$ with sufficiently small $\delta_k > 0$, we obtain from (12.23)

$$y(\tilde{t}_k) - y(\tilde{t}_k - \delta_k) + c^\star \delta_k \varepsilon_0^p \le c_4 \delta_k \sup_{t\in[\tilde{t}_k - \delta_k, \tilde{t}_k]} [\Phi(t)]^q$$

and since $y(\tilde{t}_k) - y(\tilde{t}_k - \delta_k) \ge 0$, we have

$$c^\star \varepsilon_0^p \le c_4 \sup_{t\in[\tilde{t}_k - \delta_k, \tilde{t}_k]} [\Phi(t)]^q$$

which is impossible because $\lim_\infty \Phi = 0$.

So we have proved (12.24), i.e. the first part of (12.14). The second part of (12.14) follows from (12.23) with $T_1 = T - a$, $T_2 = T + a$. If (12.16) is satisfied, too, then we obtain from (12.22), as $T_2 \to +\infty$, the first part of (12.17) and in the case when $c_3 > 0$, we find the second part of (12.17).

Finally, we obtain from (12.23) as $T_2 \to +\infty$

$$-y(T_1) + 2c_3 \int_{T_1}^\infty y(t)dt \le \int_{T_1}^\infty [\Phi(t)]^q dt.$$

Hence, by using the notation $Y(T) = \int_T^\infty y(t)dt$, we get

$$Y'(T) + 2c_3 Y(T) \le c_4 \int_T^\infty [\Phi(t)]^q dt.$$

This linear differential inequality implies (12.18) which completes the proof of Theorem 12.4. $\square$

It is easy to formulate conditions which imply that the operator $\tilde{A}(t)$, defined by (8.1), satisfies the assumptions of Theorem 12.4 in the case when $V$ is a closed linear subspace of $W^{1,p}(\Omega)$. So by Theorem 12.4 we find

**Theorem 12.5.** *Assume that the operator $\tilde{A}(t) : V \to V^\star$ satisfies the conditions of Theorem 8.1 such that with some constants $c_2 > 0$, $c_3 \geq 0$*

$$\sum_{j=0}^{n} [a_j(t, x, \xi) - a_j(t, x, \xi^\star)](\xi_j - \xi_j^\star) \geq c_2 |\xi - \xi^\star|^p + c_3 |\xi - \xi^\star|^2. \tag{12.26}$$

*Further, there exist a continuous function $\Phi$ and Carathéodory functions*

$$a_{j,\infty} : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$$

*such that for a.a. $(t, x) \in Q_\infty$, all $\xi \in \mathbb{R}^{n+1}$, $j = 0, 1, ..., n$*

$$|a_j(t, x, \xi) - a_{j,\infty}(x, \xi)| \leq \Phi(t)(|\xi|^{p-1} + 1) \text{ where } \lim_\infty \Phi = 0$$

*and there exists $F_\infty \in V^\star$ such that*

$$\|F(t) - F_\infty\|_{V^\star} \leq \Phi(t) \text{ for a.a. } t > 0.$$

*Then for a solution $u$ of (11.1) we have (12.14) where $u_\infty \in V$ is the unique solution of (12.15) with operator $A_\infty : V \to V^\star$, defined by*

$$\langle A_\infty(z), v \rangle = \sum_{j=1}^{n} \int_\Omega a_{j,\infty}(x, z, Dz) D_j v dx + \int_\Omega a_{0,\infty}(x, z, Dz) v dx. \tag{12.27}$$

*Further, (12.16) implies the first part of (12.17), if $c_3 > 0$, we have the second part of (12.17) and the estimate (12.18).*

Now we formulate and prove a stabilization result on the ("non-local") solution of (11.6), considered in Theorem 11.4.

**Theorem 12.6.** *Assume that the ("non-local") operator $A : L^p_{loc}(0, \infty; V) \to L^q_{loc}(0, \infty; V^\star)$ has the form $A(u) = B(u, u)$ where the operator*

$$B : L^p_{loc}(0, \infty; V) \times L^p_{loc}(0, \infty; V) \to L^q_{loc}(0, \infty; V^\star)$$

*is such that for each fixed $w \in L^p_{loc}(0, \infty; V)$, $[B(u, w)](t)$ depends only on $u(t)$ and this operator, mapping $V$ into $V^\star$, satisfies the assumptions of Theorem 7.1 for a.a. $t > 0$. Further, for all $u_1, u_2, w \in L^p_{loc}(0, \infty; V)$, a.a. $t > 0$*

$$\langle [B(u_1, w)](t) - [B(u_2, w)](t), u_1(t) - u_2(t) \rangle \geq \tag{12.28}$$

$$c_2 \|u_1(t) - u_2(t)\|_V^p + c_3 \|u_1(t) - u_2(t)\|_H^2$$

*with some constants $c_2 > 0$, $c_3 \geq 0$.*

*Finally, there exist $A_\infty : V \to V^\star$ and $F_\infty \in V^\star$, a continuous function $\Phi$ and for all $R > 0$ there is a positive constant $c_R$ such that for all $v \in V$ with $\|v\|_V \le R$, $w \in L^p_{loc}(0,\infty;V) \cap L^\infty(0,\infty;H)$, a.a. $t > 0$ we have*

$$\|[B(v,w)](t) - A_\infty(v)\|_{V^\star} \le c_R \Phi(t) \text{ and} \tag{12.29}$$

$$\|F(t) - F_\infty\|_{V^\star} \le \Phi(t) \tag{12.30}$$

*where* $\lim_\infty \Phi = 0$.

*Then for a solution of (11.6) we have the conclusions of Theorem 12.4, i.e. we have (12.14) and if (12.16) holds then we have (12.17), (12.18).*

*Proof.* Similarly to the proof of Theorem 12.4, one obtains that $A_\infty : V \to V^\star$ is bounded, strictly monotone, coercive and hemicontinuous. Thus the equation (12.15) has a unique solution $u_\infty$ for each $F_\infty \in V^\star$. Further, by Theorem 11.4 there exists a unique solution $u$ of (11.6). Thus one obtains

$$\langle D_t[u(t) - u_\infty], u(t) - u_\infty\rangle + \langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle = \tag{12.31}$$
$$\langle F(t) - F_\infty, u(t) - u_\infty\rangle.$$

The second term on the left hand side of (12.31) can be estimated as follows:

$$\langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle \ge c_2\|u(t) - u_\infty\|_V^p + \tag{12.32}$$
$$c_3\|u(t) - u_\infty\|_H^2 - |\langle [B(u_\infty, u)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle| \ge$$
$$c_2\|u(t) - u_\infty\|_V^p + c_3\|u(t) - u_\infty\|_H^2 -$$
$$\frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p - C(\varepsilon)\|[B(u_\infty, u)](t) - A_\infty(u_\infty)\|_{V^\star}^q.$$

For the right hand side of (12.31) we have (12.21).

Thus, choosing sufficiently small $\varepsilon > 0$, integrating (12.31) over $[T_1, T_2]$, we obtain by Remark 6.7, (12.21), (12.29), (12.30), (12.32)

$$\frac{1}{2}\|u(T_2) - u_\infty\|_H^2 - \frac{1}{2}\|u(T_1) - u_\infty\|_H^2 + \tag{12.33}$$
$$\frac{c_2}{2}\int_{T_1}^{T_2} \|u(t) - u_\infty\|_V^p dt + c_3 \int_{T_1}^{T_2} \|u(t) - u_\infty\|_H^2 dt \le \text{const} \int_{T_1}^{T_2} [\Phi(t)]^q dt.$$

Inequality (12.33) is the same as (12.22), so we can finish the proof of Theorem 12.6 as in the proof of Theorem 12.4. □

It is easy to formulate assumptions on functions

$$a_j : Q_\infty \times \mathbb{R}^{n+1} \times L^p_{loc}(0,\infty;V) \to \mathbb{R}$$

which imply that the operator $A$ of the form (11.14) satisfies the conditions of Theorem 12.6 with

$$\langle [B(u,w)](t), v\rangle = \sum_{j=1}^n \int_\Omega a_j(t,x,u,Du;w) D_j v dx +$$
$$\int_\Omega a_0(t,x,u,Du;w) v dx, \quad u, w \in L^p_{loc}(0,\infty;V), \quad v \in V.$$

**Theorem 12.7.** *Let $V$ be a closed linear subspace of $W^{1,p}(\Omega)$, $2 \le p < \infty$, $H = L^2(\Omega)$ and assume that the operator $A$ of the form (11.14) satisfies the conditions of Theorem 11.9 such that for all $u \in L^p_{loc}(0, \infty; V)$, a.a. $(t,x) \in Q_\infty$, all $\xi, \xi^\star \in \mathbb{R}^{n+1}$*

$$\sum_{j=0}^{n} [a_j(t,x,\xi;u) - a_j(t,x,\xi^\star;u)](\xi_j - \xi_j^\star) \ge$$

$$[g_2(u)](t)|\xi - \xi^\star|^p + c_3|\xi - \xi^\star|^2$$

*where $c_3$ is a nonnegative constant. Further, there exist a continuous function $\Phi$ and Carathéodory functions*

$$a_{j,\infty} : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$$

*such that for a.a. $(t,x) \in Q_\infty$, all $\xi \in \mathbb{R}^{n+1}$*

$$|a_j(t,x,\xi;u) - a_{j,\infty}(x,\xi)| \le \Phi(t)(|\xi|^{p-1} + 1), \quad j = 0, 1, ..., n$$

*where $\lim_\infty \Phi = 0$ and there exists $F_\infty \in V^\star$ such that*

$$\|F(t) - F_\infty\|_{V^\star} \le \Phi(t) \text{ for a.a. } t > 0.$$

*Then for a solution $u$ of (11.6) we have the conclusion of Theorem 12.4, i.e. (12.14) and (12.17), (12.18), respectively, where the operator $A_\infty : V \to V^\star$ is defined by (12.27).*

Now we consider Examples 10.12 and we formulate additional conditions which imply that assumptions of theorems in Sections 11 and 12 are fulfilled.

According to Example 10.12 let

$$a_j(t,x,\eta,\zeta;u) = b(t,x,[H(u)](t,x))\xi_j|\zeta|^{p-2}, \quad j = 1, ..., n,$$

$$a_0(t,x,\eta,\zeta;u) = b_0(t,x,[H_0(u)](t,x))\eta|\eta|^{p-2} +$$
$$\hat{b}_0(t,x,[F_0(u)](t,x))\hat{\alpha}_0(t,x,\eta,\zeta)$$

where $b, b_0, \hat{b}_0, \hat{\alpha}_0$ are Carathéodory functions, defined for a.a. $(t,x) \in Q_\infty$ satisfying

$$\text{const} \ge b(t,x,\theta) \ge \frac{c_2}{1 + |\theta|^{\sigma^\star}}, \quad \text{const} \ge b(t,x,\theta) \ge \frac{c_2}{1 + |\theta|^{\sigma^\star}}$$

with some constants $c_2 > 0$, $0 \le \sigma^\star < p - 1$,

$$|\hat{b}_0(t,x,\theta)| \le 1 + |\theta|^{p-1-\varrho^\star} \text{ with } 0 < \varrho^\star < p - 1 \text{ and}$$

$$|\hat{\alpha}_0(t,x,\eta,\zeta)| \le c_1(1 + |\eta|^{\hat{\varrho}} + |\zeta|^{\hat{\varrho}}), \quad \sigma^\star + \hat{\varrho} < \varrho^\star, \quad \hat{\varrho} \ge 0.$$

Further, let $H, H_0, F_0$ be operators of Volterra type such that for all $T > 0$,

$$H, H_0 : L^p(0,T;W^{1-\delta,p}(\Omega)) \to C(\overline{Q_T}), \quad F_0 : L^p(0,T;W^{1-\delta,p}(\Omega)) \to L^p(Q_T)$$

are linear continuous operators (of Volterra type).

Then clearly, the assumptions of Theorem 11.9 on existence are fulfilled and in the case when $b, b_0, \hat{b}_0$ are positive constants, the conditions of Theorem 11.5 are satisfied. If $b, b_0$ are between two positive constants, the operators $H, H_0$ may be linear continuous operators, mapping

$$L^p(0,T;W^{1-\delta,p}(\Omega)) \text{ into } L^p(Q_T), \text{ as } F_0.$$

For examples of operators of the above types, see in Example 10.12.

The conditions of Theorem 12.1 on the boundedness of $\int_\Omega u(t,x)^2 dx$ are fulfilled in the ("local") case when $b, b_0, \hat{b}_0$ are positive constants, because by Young's inequality and $\hat{\rho} < p-1$

$$|\hat{\alpha}_0(t,x,v,Dv)v| \leq \text{const}(1+|v|^{\hat{\varrho}+1}+|Dv|^{\hat{\varrho}+1}) \leq \text{const } \varepsilon^p[|v|^p+|Dv|^p]+C(\varepsilon)$$

hence with sufficiently small $\varepsilon > 0$ we obtain (12.1).

The conditions of Theorem 12.3 (on the boundedness of $\int_\Omega u(t,x)^2 dx$) are fulfilled in the "non-local" case for the above example if $H, H_0$ are linear operators of Volterra type, mapping continuously $L^2(Q_t)$ into $C(\overline{Q_t})$ for all $t > 0$. Further, $F_0$ is a linear operator of Volterra type, mapping $L^p(Q_t)$ continuously into $L^2(Q_t)$ for all $t > 0$. (If $b, b_0$ are between two positive constants, $H, H_0$ may map $L^2(Q_t)$ continuously into $L^2(Q_t)$ for all $t > 0$).

Because then

$$b(t,x,[H(u)](t,x)) \geq \frac{c_2}{1+|[H(u)](t,x)|^{\sigma^\star}} \geq \frac{c_2}{1+\|H(u)\|^{\sigma^\star}_{C(\overline{Q_t})}} \geq$$

$$\frac{c_2}{1+\text{const}\|u\|^{\sigma^\star}_{L^2(Q_t)}} \geq \frac{c_2}{1+\text{const}\left(\sup_{[0,t]}\int_\Omega u(t,x)^2 dx\right)^{\sigma^\star}}$$

and similarly can be estimated $b_0(t,x,[H_0(u)](t,x))$.

Further, by using the estimates in Example 10.12, we obtain by Young's inequality

$$|\hat{b}_0(t,x,[F_0(u)](t,x))\hat{\alpha}_0(t,x,u,Du)u| \leq$$

$$\left[1+|[F_0(u)]^{p-1-\varrho^\star}(t,x)|\right] c_1|u|(1+|u|^{\hat{\varrho}}+|Du|^{\hat{\varrho}}) \leq$$

$$\text{const}\left[1+|[F_0(u)]^{p-1-\varrho^\star}(t,x)|\right](1+|u|^{\hat{\varrho}+1}+|Du|^{\hat{\varrho}+1}) \leq$$

$$\frac{\varepsilon^p}{p}\left[|u|^{p-\sigma^\star}+|Du|^{p-\sigma^\star}\right]+C(\varepsilon)\left[1+|F_0(u)|^{q_1(p-1-\varrho^\star)}\right]$$

where

$$q_1 = \frac{p_1}{p_1-1} = \frac{p-\sigma^\star}{p-\sigma^\star-\hat{\varrho}-1}, \quad p_1 = \frac{p-\sigma^\star}{\hat{\varrho}+1} < 1$$

and

$$\int_\Omega |F_0(u)|(t,x)^{q_1(p-1-\varrho^\star)} dx \leq \text{const}\left[\sup_{\tau\in[0,t]}\int_\Omega u^2(\tau,x)dx\right]^{q_1(p-1-\varrho^\star)}.$$

Thus, choosing sufficiently small $\varepsilon > 0$, we have (12.9) with

$$[k_2(u)](t,x) = C(\varepsilon)\left[1 + |F_0(u)|^{q_1(p-1-\varrho^\star)}(t,x)\right],$$

$$\sigma = q_1(p-1-\varrho^\star) = (p-\sigma^\star)\frac{p-1-\varrho^\star}{p-1-\sigma^\star-\hat{\varrho}} < p-\sigma^\star$$

because $\sigma^\star + \hat{\varrho} < \varrho^\star$.

Finally, we formulate conditions which imply that our example satisfies the assumptions of Theorems 12.5 and 12.7, respectively (on stabilization of $u$ as $t \to \infty$). In the "local" case, when $b, b_0, \hat{b}_0$ are positive constants, assume that $\hat{\alpha}_0$ has the form

$$\hat{\alpha}_0(t,x,\eta,\zeta) = [1+\psi(t,x)]\tilde{\alpha}_0(\eta)$$

where $|\psi(t,x)| \le \Phi(t)$ and $\tilde{\alpha}_0$ is a monotone nondecreasing function, satisfying

$$|\tilde{\alpha}_0(\eta)| \le \text{const}(1+|\eta|^{\hat{\varrho}}) \text{ with } \hat{\varrho} < p-1.$$

In this case (12.26) is satisfied with $c_3 = 0$. If

$$\tilde{\alpha}_0(\eta) - \tilde{\alpha}_0(\eta^\star) \ge \tilde{c}_3(\eta-\eta^\star), \quad \tilde{c}_3 > 0$$

then we have (12.26) with some $c_3 > 0$. In this case the conclusions of Theorem 12.5 hold, assuming also

$$\|F(t) - F_\infty\| \le \Phi(t) \text{ for a.e. } t > 0$$

with some $F_\infty \in V^\star$.

In the "non-local" case assume that

$$b(t,x,\theta) = c_2[1+\psi(t,x,\theta)], \quad b_0(t,x,\theta) = \tilde{c}_2[1+\tilde{\psi}(t,x,\theta)]$$

where $c_2, \tilde{c}_2$ are positive constants and the Carathéodory functions satisfy for a.a. $(t,x) \in Q_\infty$, all $\theta \in \mathbb{R}$

$$|\psi(t,x,\theta)| \le \Phi(t), \quad |\tilde{\psi}(t,x,\theta)| \le \Phi(t) \text{ where } \sup\Phi < 1, \quad \lim_\infty \Phi = 0.$$

Further,

$$\hat{b}_0(t,x,\theta) = 1 + \psi_0(t,x,\theta) \text{ where } |\psi_0(t,x,\theta)| \le \Phi(t) \text{ and}$$

$$|\hat{\alpha}_0(t,x,\eta,\zeta)| \le [1+\psi_1(t,x)]\tilde{\beta}_0(\eta), \quad |\psi_1(t,x)| \le \Phi(t),$$

where $\tilde{\beta}_0$ is a monotone nondecreasing function, satisfying

$$|\tilde{\beta}_0(\eta)| \le \text{const}(1+|\eta|^{\hat{\varrho}}) \text{ with } \hat{\varrho} < p-1.$$

In this case (12.26) holds with $c_3 = 0$.

If

$$\tilde{\beta}_0(\eta) - \tilde{\beta}_0(\eta^\star) \ge \tilde{c}_3(\eta-\eta^\star), \quad \tilde{c}_3 > 0$$

then we have (12.26) with some $c_3 > 0$. The conclusions of Theorem 12.7 hold (with const $\Phi(t)$), assuming also

$$\|F(t) - F_\infty\|_{V^\star} \le \Phi(t), \text{ for a.a. } t > 0, \text{ with some } F_\infty \in V^\star.$$

## Problems

1. Let $u \in L^p_{loc}(0, \infty; V)$ be a solution of problem (11.1) with the operator $\tilde{A}(t)$ in Problem 4 of Section 11 and assume that $\|F(t)\|_{V^\star}$ is bounded for a.a. $t \in (0, \infty)$. Prove that $\|u(t)\|_{L^2(\Omega)}$ is bounded for $t \in (0, \infty)$ and (12.2) holds.

2. Let $u \in L^p_{loc}(0, \infty; V)$ be a solution of problem (11.1) with the operator $\tilde{A}(t)$, $t \in (0, \infty)$ in Problem 3 of Section 8 (see also Problem 5 in Section 11) satisfying the conditions
$$c_1 \le \alpha(t, x) \le c_2, \quad c_1 \le \beta(t, x) \le c_2 \text{ for a.a. } (t, x) \in Q_\infty$$
with some positive constants $c_1, c_2$. Assume that $F \in L^\infty(0, \infty; V)$. Prove that $\|u(t)\|_{L^2(\Omega)}$ is bounded for $t \in (0, \infty)$ and (12.2) holds.

3. Let $u \in L^p_{loc}(0, \infty; V)$ be a solution of problem (11.1) with the operator $\tilde{A}(t)$, $t \in (0, \infty)$ in Problem 6 of Section 8 (see also Problem 6 in Section 11). Assuming $F \in L^\infty(0, \infty; V)$, prove that $\|u(t)\|_{L^2(\Omega)}$ is bounded for $t \in (0, \infty)$ and (12.2) holds.

4. Assume that the operator $\tilde{A}(t)$, defined in Problem 4 of Section 10 satisfies the conditions in that Problem for all $t \in (0, \infty)$. Prove that if $u$ is a solution of problem (11.1) with the above operator $\tilde{A}(t)$ and $F \in L^\infty(0, \infty; V)$ then $\|u(t)\|_{L^2(\Omega)}$ is bounded for $t \in (0, \infty)$ and (12.2) holds.

5. Assume that $u \in L^p_{loc}(0, \infty; V)$ is a solution of problem (11.1) with the operator $\tilde{A}(t)$, $t \in (0, \infty)$ considered in Problem 3 of Section 8 (for $0 \le t \le T$) and in Problem 2 for $t \in (0, \infty)$. Further, assume that there exist functions $\alpha_\infty, \beta_\infty \in L^\infty(\Omega)$ such that
$$\lim_{t\to\infty} \|\alpha(t, \cdot) - \alpha_\infty\|_{L^\infty(\Omega)} = 0, \quad \lim_{t\to\infty} \|\beta(t, \cdot) - \beta_\infty\|_{L^\infty(\Omega)} = 0.$$
Further, there exists $F_\infty \in V^\star$ such that
$$\lim_{t\to\infty} \|F(t) - F_\infty\|_{V^\star} = 0.$$
Prove that then
$$\lim_{t\to\infty} \|u(t) - u_\infty\|_{L^2(\Omega)} = 0, \quad \lim_{T\to\infty} \int_{T-a}^{T+a} \|u(t) - u_\infty\|_V^p dt = 0$$
for arbitrary fixed $a > 0$ where $u_\infty$ is the unique solution to
$$\int_\Omega \left[\alpha_\infty(x)(D_j u_\infty)|Du_\infty|^{p-2} D_j w + \beta_\infty(x) u_\infty |u_\infty|^{p-2} w\right] dx =$$
$$\langle F_\infty, w\rangle, \quad w \in V.$$

6. Formulate and prove a theorem on the stabilization of the solution of (11.1) (as $t \to \infty$) with the operator $\tilde{A}$ defined by

$$\langle [\tilde{A}(t)][u(t)], v(t)\rangle = \sum_{|\alpha|\le m} \int_\Omega (D^\alpha u)|D^\alpha u|^{p-2} D^\alpha v dx, \quad t \in (0,\infty),$$

$u, v \in L^p_{loc}(0,\infty; V)$, $V$ is a closed linear subspace of $W^{m,p}(\Omega)$, if there exists $F_\infty \in V^\star$ such that

$$\lim_{t\to\infty} \|F(t) - F_\infty\|_{L^\infty(\Omega)} = 0.$$

# 13 Periodic solutions

In this section we shall formulate conditions which imply the existence of $T$-periodic solutions of evolution equations in $(0,\infty)$. In the proofs we shall apply the following maximal monotone operator. (See (9.11), Remark 9.5.)

**Definition 13.1.** *Let $V \subset H \subset V^\star$ be an evolution triple and define operator $L$ by*

$$Lu = u', \quad D(L) = \{u \in L^p(0,T;V) : u' \in L^q(0,T;V^\star), u(0) = u(T)\} \tag{13.1}$$

**Theorem 13.2.** *The operator (13.1) is a closed, linear, densely defined, maximal monotone mapping from $L^p(0,T;V)$ into $L^q(0,T;V^\star)$. The* maximal monotonicity of $L$ *means that it is monotone and it has no proper monotone extension.*

*Proof.* It is not difficult to show that $L$ is a closed, linear, densely defined operator, mapping from $L^p(0,T;V)$ into $L^q(0,T;V^\star)$. Further, the operator $L$ is monotone because by Remark 6.7 for arbitrary $u \in D(L)$

$$[Lu, u] = \int_0^T \langle u'(t), u(t)\rangle dt = \frac{1}{2}\left[\|u(T)\|_H^2 - \|u(0)\|_H^2\right] = 0.$$

Further, assume that for some $v \in L^p(0,T;V)$, $w \in L^q(0,T;V^\star)$

$$[w - Lu, v - u] \ge 0 \text{ for all } u \in D(L). \tag{13.2}$$

We have to show that $v \in D(L)$ and $w = Lv = v'$. Apply (13.2) to $u(t) = \lambda\psi(t)z$ where $z \in V$, $\psi \in C_0^\infty(0,T)$ and $\lambda \in \mathbb{R}$ are arbitrary. Since

$$[Lu, u] = \int_0^T \langle u'(t), u(t)\rangle dt = \int_0^T \lambda^2 \psi'(t)\langle z, z\rangle dt = 0,$$

we obtain from (13.2)

$$[w, v - u] - [Lu, v] \ge 0, \text{ i.e.}$$

$$\int_0^T \langle w(t), v(t)\rangle dt - \lambda \left[\int_0^T \langle w(t), \psi(t)z\rangle dt - \int_0^T \langle \psi'(t)z, v(t)\rangle dt\right] \geq 0.$$

This inequality may hold for arbitrary $\lambda \in \mathbb{R}$, only if

$$\int_0^T \langle w(t), \psi(t)z\rangle dt + \int_0^T \langle \psi'(t)z, v(t)\rangle dt = 0$$

which implies according to Remark 6.5 that

$$v' = w \in L^q(0,T;V^\star).$$

Further, by using the formula in Remark 6.7, we obtain from (13.2) and $u(0) = u(T)$

$$0 \leq 2[v' - u', v - u] = \|v(T) - u(T)\|_H^2 - \|v(0) - u(0)\|_H^2 = \tag{13.3}$$

$$(v(T), v(T)) + (u(T), u(T)) - 2(u(T), v(T)) - (v(0), v(0)) - (u(0), u(0)) +$$
$$2(u(0), v(0)) = \|v(T)\|_H^2 - \|v(0)\|_H^2 + 2(u(0), v(0) - v(T)).$$

The inequality (13.3) implies $v(0) = v(T)$, i.e. $v \in D(L)$. Indeed, assuming $v(0) \neq v(T)$, one could find $u \in D(L)$ such that the right hand side of (13.3) would be negative, since for arbitrary $v \in V$, the function

$$u(t) = v, \quad t \in [0,T]$$

belongs to $D(L)$. So we have shown that $L$ is maximal monotone. □

Now consider evolution equations in $(0,\infty)$ with "local" operators $A$ which have the form

$$[A(u)](t) = [\tilde{A}(t)][u(t)]. \tag{13.4}$$

**Theorem 13.3.** *Assume that for a.a. $t > 0$, the function $t \mapsto \tilde{A}(t)$ is $T$-periodic (i.e. $\tilde{A}(t+T) = \tilde{A}(t)$ for a.a. $t > 0$), and satisfies the conditions of Theorem 9.2, further, $F \in L^q_{loc}(0,\infty;V^\star)$ is $T$-periodic, too.*

*Then there exists a $T$-periodic function $u \in L^p_{loc}(0,\infty;V)$ which satisfies $u' \in L^q_{loc}(0,\infty;V^\star)$ and*

$$u'(t) + [\tilde{A}(t)][u(t)] = F(t) \text{ for a.a. } t > 0. \tag{13.5}$$

*Proof.* The assumptions of Theorem 13.3 imply that the operator

$$A : L^p(0,T;V) \to L^q(0,T;V^\star),$$

, defined by

$$[A(u)](t) = [\tilde{A}(t)][u(t)], \quad t \in [0,T]$$

is bounded, coercive and pseudomonotone with respect to $W^1_p(0,T;V,H)$, and, consequently, it is pseudomonotone with respect to $D(L)$ (defined by (13.1)).

Further, we claim that it is demicontinuous. Indeed, for a.a. fixed $t$, $\tilde{A}(t) : V \to V^\star$ is demicontinuous, thus, if $(u_k) \to u$ with respect to the norm of $L^p(0,T;V)$ then for a.a. $t \in [0,T]$, $(u_k(t)) \to u(t)$ with respect to the norm of $V$ (for a subsequence) which implies that for each fixed $v \in L^p(0,T;V)$, a.a. $t \in [0,T]$

$$\langle [\tilde{A}(t)][u_k(t)], v(t)\rangle \to \langle [\tilde{A}(t)][u(t)], v(t)\rangle \text{ as } k \to \infty$$

(since $\tilde{A}(t)$ is demicontinuous), so Vitali's theorem, Hölder's inequality and the boundedness assumption (9.5) imply

$$[A(u_k), v] \to [A(u), v] \text{ as } k \to \infty.$$

Thus by Theorems 9.4, 13.2 there exists a solution $u \in D(L)$ of (13.5) in $[0,T]$. Since $u \in D(L)$, we have $u(0) = u(T)$. Thus, defining $u(t)$ for $t \geq 0$ by

$$u(t + kT) = u(t), \quad t \in [0,T], \quad k = 1, 2, ...$$

we obtain

$$u \in L^p_{loc}(0,\infty;V), \quad u' \in L^q_{loc}(0,\infty;V^\star) \text{ and } u \text{ is } T\text{-periodic.}$$

($u' \in L^q_{loc}(0,\infty;V^\star)$ follows from $u(0) = u(T)$ and formula (6.9).) Thus $u$ satisfies (13.5) in $(0,\infty)$. □

Applying Theorem 13.3 in the case when $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, $p \geq 2$, $H = L^2(\Omega)$, to operators of the form (8.1), we obtain directly

**Theorem 13.4.** *Assume that the functions $a_j : Q_\infty \times \mathbb{R}^{n+1} \to \mathbb{R}$ are $T$-periodic, i.e. for a.a. $t > 0, x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$*

$$a_j(t + T, x, \xi) = a_j(t, x, \xi)$$

*and their restrictions to $[0,T]$ satisfy (B1), (B2), ($\tilde{B}3$) or (B3$'$) and (B4). Further, $F \in L^q_{loc}(0,\infty;V^\star)$ is $T$-periodic, too.*

*Then there exists a $T$-periodic solution $u \in L^p_{loc}(0,\infty;V)$ of (13.5) where the operator $\tilde{A}(t) : V \to V^\star$ is defined by (8.1).*

In the case of "non-local" operators $A$, instead of the abstract Cauchy problem we consider the following modified problem, which is a generalization of the Cauchy problem for functional differential equations in one variable (see Remark 10.11):

$$u'(t) + \hat{A}(t, u_t) = F(t) \text{ for a.a. } t \in [0,\infty], \quad u(t) = \psi(t), \text{ for a.a. } t \in [-a, 0] \tag{13.6}$$

where $u_t$ is defined by

$$u_t(s) = u(t + s), \quad s \in [-a, 0], \quad t \geq 0 \tag{13.7}$$

$\psi \in L^p(-a, 0; V)$, $F \in L^q_{loc}(0,\infty;V^\star)$ are given functions and we want to find a function $u \in L^p_{loc}(-a,\infty;V)$ such that $u' \in L^q_{loc}(0,\infty;V^\star)$ and $u$ satisfies (13.6). Further,

$$\hat{A} : (0,\infty) \times L^p(-a, 0; V) \to L^q_{loc}(0,\infty;V^\star) \tag{13.8}$$

is a given (nonlinear) operator. Observe that defining operator

$$A : L^p_{loc}(-a, \infty; V) \to L^q_{loc}(0, \infty; V^\star) \text{ by}$$

$$[A(u)](t) = \hat{A}(t, u_t), \quad u \in L^p_{loc}(-a, \infty; V), \quad t > 0$$

the differential equation in (13.6), i.e.

$$u'(t) + [A(u)](t) = F(t), \quad t > 0, \quad u' \in L^q_{loc}(0, \infty; V^\star)$$

has the form (11.6) which was considered in Section 11. We assume that $A$ is of Volterra type and $[A(u)](t)$ depends only on $u\mid_{[t,t-a]}$.

We shall formulate conditions on $\hat{A}$ and $F$ which imply that for some $\psi \in L^p(-a, 0; V)$ there exists a $T$-periodic solution of problem (13.6).

**Theorem 13.5.** *Assume that the operator*

$$\hat{A} : (0, \infty) \times L^p(-a, 0; V) \to L^q_{loc}(0, \infty; V^\star)$$

*and $F$ are $T$-periodic, i.e. for all $v \in L^p(-a, 0; V)$*

$$\hat{A}(t + T, v) = \hat{A}(t, v), \quad F(t + T) = F(t) \text{ for a.a. } t \in (0, \infty),$$

*and $\hat{A}$ is of Volterra type. Further, assume that the operator $\tilde{A} : L^p(0, T; V) \to L^q(0, T; V^\star)$, defined by*

$$[\tilde{A}(u)](t) = \hat{A}(t, (Pu)_t), \quad t \in [0, T], \quad u \in L^p(0, T; V) \tag{13.9}$$

$$(Pu)(t) = u(t + kT) \text{ if } t > -a \text{ and } t + kT \in (0, T) \text{ for some } k = 0, 1, 2, ..., \tag{13.10}$$

*is bounded, demicontinuous, coercive and pseudomonotone with respect to*

$$D(L) = \{u \in L^p(0, T; V) : u' \in L^q(0, T; V^\star), \quad u(T) = u(0)\}. \tag{13.11}$$

*Then there exists $u \in L^p_{loc}(-a, \infty; V)$ such that $u' \in L^q_{loc}(-a, \infty; V^\star)$,*

$$u'(t) + \hat{A}(t, u_t) = F(t), \quad u(t + T) = u(t) \text{ for a.a. } t \in (0, \infty).$$

**Remark 13.6.** Theorem 13.5 means that for all $T > 0$ there exists

$$\psi \in L^p(-a, 0; V) \text{ with } \psi' \in L^q(-a, 0; V^\star)$$

such that there exists a $T$-periodic solution of the Cauchy problem (13.6).

*Proof of Theorem 13.5.* Since by Theorem 13.2 $L = D_t$ is a maximal monotone, closed, densely defined linear operator with $D(L)$, given in (13.11) and $\tilde{A} : L^p(0, T; V) \to L^q(0, T; V^\star)$ is bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$, by Theorem 9.4 there is a solution $u \in D(L)$ of

$$u' + \tilde{A}(u) = F.$$

Then for $\tilde{P}u$, defined by

$$(\tilde{P}u)(t) = u(t + kT), \quad t > -a \text{ and } t + kT \in [0, T] \text{ for some integer } k$$

we have $\tilde{P}u \in L^p_{loc}(-a, \infty; V)$, $\tilde{P}u$ is $T$-periodic, $(\tilde{P}u)' \in L^q_{loc}(-a, \infty; V^\star)$ and satisfies

$$(\tilde{P}u)'(t) + \hat{A}(t, (\tilde{P}u)_t) = F(t), \text{ for a.a. } t \in (0, \infty),$$

i.e. the statement of Theorem 13.5 holds for $\tilde{P}u$. □

Now we apply Theorem 13.5 in the particular case when $\hat{A}$ has the form analogous to the formula (10.37) and $V$ is a closed linear subspace of $W^{1,p}(\Omega)$, $H = L^2(\Omega)$. Similarly to the conditions $(C1)$ – $(C5)$ and the conditions of Theorem 11.9, assume

$(C1^\star)$ The functions

$$a_j : Q_\infty \times \mathbb{R}^{n+1} \times L^p(-a, 0; V) \to \mathbb{R}$$

satisfy the Carathéodory conditions for arbitrary fixed $w \in L^p(-a, 0; V)$ $(j = 0, 1, ..., n)$), and are $T$-periodic:

$$a_j(t + T, x, \xi; w) = a_j(t, x, \xi; w)$$

for a.e. $(t, x) \in Q_\infty$, all $\xi \in \mathbb{R}^{n+1}$, $w \in L^p(-a, 0; V)$.

$(C2^\star)$ There exist bounded (nonlinear) operators $g_1 : L^p(-a, 0; V) \to \mathbb{R}^+$ and $k_1 : L^p(-a, 0; V) \to L^q(Q_T)$ such that

$$|a_j(t, x, \eta, \zeta; w)| \leq g_1(w)[1 + |\eta|^{p-1} + |\zeta|^{p-1}] + [k_1(w)](t, x)$$

for a.e. $(t, x) \in Q_T$, each $(\eta, \zeta) \in \mathbb{R}^{n+1}$ and $w \in L^p(-a, 0; V)$.

$(C3^\star)$ There holds the inequality

$$\sum_{j=1}^{n} [a_j(t, x, \eta, \zeta; w) - a_j(t, x, \eta, \zeta^\star; w)](\xi_j - \xi_j^\star) \geq [g_2(w)]|\zeta - \zeta^\star|^p$$

where

$$g_2(w) \geq c^\star \left[1 + \|w\|_{L^p(-a,0;V)}\right]^{-\sigma^\star},$$

$c^\star$ is some positive constant and $0 \leq \sigma^\star < p - 1$.

$(C4^\star)$ There holds the inequality

$$\sum_{j=0}^{n} a_j(t, x, \eta, \zeta; w)\xi_j \geq [g_2(w)][1 + |\eta|^p + |\zeta|^p] - [k_2(w)](t, x)$$

where $k_2(w) \in L^1(Q_T)$ satisfies for some positive $\sigma < p - \sigma^\star$

$$\|k_2(w)\|_{L^1(Q_T)} \leq \text{const} \left[1 + \|w\|_{L^p(-a,0;V)}\right]^\sigma.$$

$(C5^\star)$ There exists $\delta > 0$ such that if $(w_k) \to w$ in $L^p(-a, 0; V)$, strongly in $L^p(-a, 0; W^{1-\delta}(\Omega))$, $(\eta^k) \to \eta$ in $\mathbb{R}$, $(\zeta^k) \to \zeta$ in $\mathbb{R}^n$, then for a.a. $(t, x) \in Q_T$, $j = 0, 1, ..., n$,

$$\lim_{k\to\infty} a_j(t, x, \eta^k, \zeta^k; w_k) = a_j(t, x, \eta, \zeta; w),$$

for a subsequence.

**Definition 13.7.** *Assuming (C1⋆) – (C5⋆), we define operator*

$$A : L^p_{loc}(-a, \infty; V) \to L^q_{loc}(0, \infty; V^\star) \text{ by}$$

$$\langle [A(u)](t), v\rangle = \int_\Omega \left\{ \sum_{j=1}^n a_j(t, x, u, Du; u_t) D_j v + a_0(t, x, u, Du; u_t) v \right\} dx,$$
$$u \in L^p_{loc}(-a, \infty; V), \quad v \in V. \tag{13.12}$$

**Theorem 13.8.** *Let $V$ be a closed linear subspace of $W^{1,p}(\Omega)$, $H = L^2(\Omega)$, assume (C1⋆) – (C5⋆) and let $F \in L^q_{loc}(0, \infty; V^\star)$ be $T$-periodic. Then there exists $u \in L^p_{loc}(-a, \infty; V)$ such that $u' \in L^q_{loc}(-a, \infty; V^\star)$ and*

$$u'(t) + [A(u)](t) = F(t), \quad u(t + T) = u(t) \text{ for a.a. } t \in (0, \infty).$$

*Proof.* Let $\hat{A}(t, u_t) = [A(u)](t)$ where $[A(u)](t)$ is given by (13.12), then the operator

$$\tilde{A} : L^p(0, T; V) \to L^q(0, T; V^\star),$$

given in (13.9), has the form

$$\langle [\tilde{A}(u)](t), v\rangle = \langle \hat{A}(t, (Pu)_t), v\rangle =$$
$$\int_\Omega \left\{ \sum_{j=1}^n a_j(t, x, u, Du; (Pu)_t) D_j v + a_0(t, x, u, Du; (Pu)_t) v \right\} dx,$$
$$u \in L^p(0, T; V), \quad v \in V$$

where $Pu$ is defined by (13.10). By Theorem 10.9 the assumptions $(C1^\star)$, $(C2^\star)$, $(C4^\star)$ imply that $\tilde{A}$ is bounded, demicontinuous and coercive. Further, $(C1^\star)$ – $(C5^\star)$ imply that $\tilde{A}$ is pseudomonotone with respect to $D(L)$, given by (13.11). This statement can be proved by using the arguments of the proof of Theorem 10.9. Thus Theorem 13.8 directly follows from Theorem 13.5. □

Now we formulate conditions which imply that the Examples 10.12 satisfy the conditions of Theorem 13.8.

**Example 13.9.** Assume that the functions $b, b_0, \hat{b}_0, \hat{\alpha}_0$ are $T$-periodic. Further, operators $H, H_0, F_0$ have the form

$$H(u) = \tilde{H}(u_t), \quad H_0(u) = \tilde{H}_0(u_t), \quad F_0(u) = \tilde{F}_0(u_t)$$

where

$$\tilde{H}, \tilde{H}_0 : L^p(-a, 0; W^{1-\delta}(\Omega)) \to C(\overline{Q_T}), \quad \tilde{F}_0 : L^p(-a, 0; W^{1-\delta}(\Omega)) \to L^p(Q_T)$$

are linear continuous operators. Then the conditions of Theorem 13.8 on $a_j$ are fulfilled.

## Problems

1. Show that for the Example 13.9 the assumptions of Theorem 13.8 are fulfilled.

2. Consider the functions

$$\alpha_j : Q_\infty \times \mathbb{R} \to \mathbb{R}, \quad j = 0, 1, \dots, n$$

   which satisfy the assumptions of Problem 1 in Section 8 for all $t \in (0, \infty)$ (see also Problem 4 in Section 11) and

$$\alpha_j(t + T, x, \xi_j) = \alpha_j(t, x, \xi_j) \text{ for a.a. } t \geq 0, \quad x \in \Omega, \quad \xi_j \in \mathbb{R}.$$

   Further, $F \in L^q_{loc}(0, \infty; V^\star)$ satisfies

$$F(t + T) = F(t) \text{ for a.a. } t \geq 0.$$

   Prove that there exists a $T$-periodic solution $u \in L^p(0, \infty; V)$ of the equation (13.5) with the operator $\tilde{A}(t)$ defined by functions $\alpha_j$ in Problem 1 of Section 8.

3. Formulate and prove a theorem on the existence of a $T$-periodic solution of the equation (13.5) where the operator $\tilde{A}(t)$ is defined in Problem 3 of Section 8.

4. Formulate and prove a theorem on the existence of a $T$-periodic solution of the equation (13.5) where the operator $\tilde{A}(t)$ is defined in Problem 6 of Section 8.

5. Formulate and prove a theorem on the existence of a $T$-periodic solution of the equation (13.5) where the operator $\tilde{A}(t)$ is defined in Problem 4 of Section 10.

# Chapter 3

# SECOND ORDER EVOLUTION EQUATIONS

In this chapter we shall consider certain nonlinear hyperbolic differential equations and functional equations which can be treated by means of monotone type operators. Namely, we shall consider equations of the form

$$u'' + N(u') + Qu + M(u', u) = F$$

where $N$ is a nonlinear operator of monotone type, $Q$ is a linear operator having some particular properties and $M$ is a nonlinear operator with some compactness properties, finally, $F \in L^q(0,T;V^\star)$.

## 14 Existence of solutions in $(0,T)$

As before, let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$ and let the operator $L$ be defined by

$$Lu = u', \quad D(L) = \{u \in L^p(0,T;V) : u' \in L^q(0,T;V^\star), \quad u(0) = 0\}.$$

Assume that

$(D1)$ $N : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, pseudomonotone with respect to $D(L)$ and coercive such that

$$[N(v), v] = \int_0^T \langle [N(v)](t), v(t)\rangle dt \geq c_2 \|v\|^p_{L^p(0,T;V)} - c_3, \quad v \in L^p(0,T;V)$$

with some constants $c_2 > 0, c_3$.

$(D2)$ $\tilde{Q} : V \to V^\star$ is a linear continuous operator with the properties:

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \langle \tilde{Q}\tilde{v}, \tilde{u}\rangle, \quad \langle \tilde{Q}\tilde{u}, \tilde{u}\rangle \geq 0 \text{ for any } \tilde{u}, \tilde{v} \in V$$

and define $Q : L^p(0,T;V) \to L^q(0,T;V^\star)$ by

$$(Qu)(t) = \tilde{Q}u(t), \quad u \in L^p(0,T;V).$$

$(D3)$ The operator

$$M : L^p(0,T;V) \times L^p(0,T;V) \to L^q(0,T;V^\star)$$

is bounded, demicontinuous, it has the following compactness property: if $(u_k) \to u$ weakly in $L^p(0,T;V)$, $(u_k') \to u'$ weakly in $L^p(0,T;V)$ and $(u_k'') \to u''$ weakly in $L^q(0,T;V^\star)$ then for a subsequence

$$(M(u_k', u_k)) \to M(u', u) \text{ weakly in } L^q(0,T;V^\star) \text{ and}$$

$$\lim_{k\to\infty} \int_0^T \langle M(u_k', u_k)(t), u_k'(t) - u'(t)\rangle dt = 0.$$

Finally,

$$\lim_{\|(u,v)\|\to\infty} \frac{\|M(v,u)\|^p_{L^q(0,TV^\star)}}{\|u\|^p_{L^p(0,T;V)} + \|v\|^p_{L^p(0,T;V)}} = 0$$

where $\|(u,v)\| = \|u\|_{L^p(0,T;V)} + \|v\|_{L^p(0,T;V)}$.

**Theorem 14.1.** *Assume (D1) – (D3). Then for arbitrary $F \in L^q(0,T;V^\star)$ there exists $u \in C^1([0,T];H) \cap C([0,T];V)$ such that $u' \in L^p(0,T;V)$, $u'' \in L^q(0,T;V^\star)$ and*

$$u'' + N(u') + Qu + M(u', u) = F \text{ in } [0,T], \tag{14.1}$$

$$u(0) = 0, \quad u'(0) = 0. \tag{14.2}$$

*Proof.* Define operator $S : L^p(0,T;V) \to C([0,T];V)$ by

$$(Sv)(t) = \int_0^t v(s)ds.$$

Clearly, $S$ is a linear and continuous operator. If $u$ is a solution of (14.1), (14.2) then $v = u'$ satisfies $v \in L^p(0,T;V)$, $v' \in L^q(0,T;V^\star)$ and

$$v' + N(v) + QSv + M(v, Sv) = F \tag{14.3}$$

$$v(0) = 0. \tag{14.4}$$

Further, if $v \in L^p(0,T;V)$ satisfies (14.3), (14.4) then $u = Sv$ is a solution of (14.1), (14.2), since $u = Sv$ is absolutely continuous and $u'(t) = v(t)$ for a.a. $t \in [0,T]$. Thus, due to Theorem 9.4, it is sufficient to show that the operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$, defined by

$$A(v) = N(v) + QSv + M(v, Sv)$$

is bounded, demicontinuous, pseudomonotone with respect to $D(L)$ and it is coercive.

Since the operator $S : L^p(0,T;V) \to L^p(0,T;V)$ is linear and continuous, from assumptions (i) - (iii) directly follows that $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded and demicontinuous.

Now we show that $A$ is pseudomonotone with respect to $D(L)$. Let $(v_k)$ be a sequence in $D(L)$ such that

$$(v_k) \to v \text{ weakly in } L^p(0,T;V), \quad (v_k') \to v' \text{ weakly in } L^q(0,T;V^\star), \tag{14.5}$$

$$\limsup_{k\to\infty}[A(v_k), v_k - v] \leq 0. \tag{14.6}$$

By (ii) the linear operator $QS : L^p(0,T;V) \to L^q(0,T;V^\star)$ is monotone. Indeed, by using the notation $u = Sv$, we have $v = u'$ and thus

$$[QSv, v] = [Qu, u'], \tag{14.7}$$

so

$$[Qu, u'] = \int_0^T \langle \tilde{Q}u(t), u'(t)\rangle dt = \frac{1}{2}\langle \tilde{Q}u(T), u(T)\rangle - \frac{1}{2}\langle \tilde{Q}u(0), u(0)\rangle = \tag{14.8}$$

$$\frac{1}{2}\langle \tilde{Q}u(T), u(T)\rangle \geq 0.$$

To obtain formula (14.8) we choose a sequence of polynomials $q_l : [0,T] \to V$ such that

$$q_l' \to u' \text{ in } W_p^1(0,T;V,H), \quad q_l \to u \text{ in } C([0,T];V) \text{ as } l \to \infty.$$

Then

$$\langle \tilde{Q}q_l(t), q_l(t)\rangle' = \langle \tilde{Q}q_l'(t), q_l(t)\rangle + \langle \tilde{Q}q_l(t), q_l'(t)\rangle =$$

$$2\langle \tilde{Q}q_l(t), q_l'(t)\rangle,$$

and after integrating over $[0,T]$ we obtain

$$\int_0^T \langle \tilde{Q}q_l(t), q_l'(t)\rangle dt = \frac{1}{2}\langle \tilde{Q}q_l(T), q_l(T)\rangle - \frac{1}{2}\langle \tilde{Q}q_l(0), q_l(0)\rangle$$

and so (14.8) follows as $l \to \infty$.

Consequently,

$$[QSv_k - QSv, v_k - v] \geq 0,$$

hence

$$[QSv_k, v_k - v] \geq [QSv, v_k - v] \to 0 \text{ as } k \to \infty,$$

which implies

$$\liminf_{k\to\infty}[QSv_k, v_k - v] \geq 0. \tag{14.9}$$

Set $u_k = Sv_k$, $u = Sv$ then $v_k = u_k'$, $v = u'$ and

$$(Sv_k) \to Sv \text{ weakly in } L^p(0,T;V), \text{ i.e. } u_k \to u \text{ weakly in } L^p(0,T;V) \quad (14.10)$$

and by (14.5)

$$(u_k') \to u' \text{ weakly in } L^p(0,T;V), \quad (u_k'') \to u'' \text{ weakly in } L^q(0,T;V^\star). \quad (14.11)$$

Thus by assumption $(D3)$ for a subsequence (denoted in the same way) we obtain

$$M(v_k, Sv_k) \to M(v, Sv) \text{ weakly in } L^q(0,T;V^\star), \quad (14.12)$$

$$\lim_{k\to\infty} [M(v_k, Sv_k), v_k - v] = 0. \quad (14.13)$$

Now (14.6), (14.9), (14.13) imply

$$\limsup_{k\to\infty} [N(v_k), v_k - v] \le 0, \quad (14.14)$$

for a subsequence. By using Cantor's trick one obtains that (14.14) holds for the original sequence, too.

Since according to $(D1)$ $N$ is pseudomonotone with respect to $D(L)$, by (14.5), (14.14) we have

$$(N(v_k)) \to N(v) \text{ weakly in } L^q(0,T;V^\star), \quad (14.15)$$

$$\lim_{k\to\infty} [N(v_k), v_k - v] = 0. \quad (14.16)$$

$>$From (14.6), (14.13), (14.16) one gets

$$\limsup_{k\to\infty} [QS(v_k), v_k - v] \le 0$$

for a subsequence and so by (14.9)

$$\lim_{k\to\infty} [QS(v_k), v_k - v] = 0, \quad (14.17)$$

whence

$$\lim_{k\to\infty} [A(v_k), v_k - v] = 0 \quad (14.18)$$

for a subsequence, thus by using Cantor's trick we find (14.18) for the original sequence, too.

Since $QS : L^p(0,T;V) \to L^q(0,T;V^\star)$ is linear, continuous and monotone, by Proposition 2.5 it is pseudomonotone which implies by (14.17)

$$(QS(v_k)) \to QS(v) \text{ weakly in } L^q(0,T;V^\star). \quad (14.19)$$

Therefore, (14.12), (14.15), (14.19) imply

$$(A(v_k)) \to A(v) \text{ weakly in } L^q(0,T;V^\star)$$

(for a subsequence), so by (14.18) we have shown that $A$ is pseudomonotone with respect to $D(L)$.

Finally, we prove that $A$ is coercive. By assumption (i) and the monotonicity of $QS$

$$\frac{[A(v),v]}{\|v\|^p_{L^p(0,T;V)}} \geq c_2 - \frac{c_3}{\|v\|^p_{L^p(0,T;V)}} - \frac{|[M(v,Sv),v]|}{\|v\|^p_{L^p(0,T;V)}} \tag{14.20}$$

and for the last term we have

$$\frac{|[M(v,Sv),v]|}{\|v\|^p_{L^p(0,T;V)}} \leq \left[\frac{\|M(v,Sv)\|^q_{L^q(0,T;V^\star)}}{\|v\|^p_{L^p(0,T;V)}}\right]^{1/q}, \tag{14.21}$$

$$\frac{\|M(v,Sv)\|^q_{L^q(0,T;V^\star)}}{\|v\|^p_{L^p(0,T;V)}} = \tag{14.22}$$

$$\frac{\|M(v,Sv)\|^q_{L^q(0,T;V^\star)}}{\|v\|^p_{L^p(0,T;V)} + \|Sv\|^p_{L^p(0,T;V)}} \times \frac{\|v\|^p_{L^p(0,T;V)} + \|Sv\|^p_{L^p(0,T;V)}}{\|v\|^p_{L^p(0,T;V)}}.$$

According to assumption $(D3)$, for arbitrary $\varepsilon > 0$ there exists $a > 0$ such that

$$\|v\|^p_{L^p(0,T;V)} + \|Sv\|^p_{L^p(0,T;V)} > a \text{ implies } \frac{\|M(v,Sv)\|^q_{L^q(0,T;V^\star)}}{\|v\|^p_{L^p(0,T;V)} + \|Sv\|^p_{L^p(0,T;V)}} < \varepsilon.$$

Thus by the boundedness of $S$ and (14.22)

$$\frac{\|M(v,Sv)\|^q_{L^q(0,T;V^\star)}}{\|v\|^p_{L^p(0,T;V)}} \leq C\varepsilon + \frac{1}{\|v\|^p_{L^p(0,T;V)}} \sup_{\|v\|^p + \|u\|^p \leq a} \|M(v,u)\|^q_{L^q(0,T;V^\star)} =$$

$$C\varepsilon + \frac{C^\star(a)}{\|v\|^p_{L^p(0,T;V)}}$$

with some constant $C > 0$ and a constant $C^\star > 0$, depending on $a$. Choosing sufficiently small $\varepsilon > 0$, we obtain

$$\frac{\|M(v,Sv)\|^q_{L^q(0,T;V^\star)}}{\|v\|^p_{L^p(0,T;V)}} \leq (c_2/2)^q$$

if $\|v\|_{L^p(0,T;V)}$ is sufficiently large, whence by (14.20), (14.21) we find

$$\frac{[A(v),v]}{\|v\|^p_{L^p(0,T;V)}} \geq c_2/2 - \frac{c_3}{\|v\|^p_{L^p(0,T;V)}}$$

if $\|v\|_{L^p(0,T;V)}$ is sufficiently large. Thus, $A$ is coercive which completes the proof of Theorem 14.1. $\square$

Now assume that instead of $(D1)$ the following (stronger) condition is fulfilled.

$(D1')$ $N : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, coercive (as in $(D1)$) and is of $(S)_+$ with respect to $D(L)$ (see, e.g. [8], [93]): if for $v_k \in D(L)$

$$(v_k) \to v \text{ weakly in } L^p(0,T;V), \quad (v_k') \to v' \text{ weakly in } L^q(0,T;V^\star),$$

$$\limsup_{k\to\infty}[N(v_k), v_k - v] \le 0 \text{ then } (v_k) \to v \text{ in } L^p(0,T;V).$$

(Then, clearly, $N$ is pseudomonotone with respect to $D(L)$.)

In this case we may assume a weaker condition on $M$:

$(D3')$ The operator $M : L^p(0,T;V)\times L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous. Further,

$$\text{if } (u_k) \to u, \quad (u_k') \to u' \text{ weakly in } L^p(0,T;V),$$

$$(u_k'') \to u'' \text{ weakly in } L^q(0,T;V^\star)$$

then for a subsequence

$$\lim_{k\to\infty}\int_0^T \langle [M(u_k', u_k)](t), u_k'(t) - u'(t)\rangle dt = 0.$$

Finally,

$$\lim_{\|(u,v)\|\to\infty} \frac{\|M(v,u)\|^q_{L^q(0,T;V^\star)}}{\|u\|^p_{L^p(0,T;V)} + \|v\|^p_{L^p(0,T;V)}} = 0. \tag{14.23}$$

**Theorem 14.2.** *Assume $(D1')$, $(D2)$, $(D3')$. Then for arbitrary $F \in L^q(0,T;V^\star)$ there exists $u \in C([0,T];V)$ such that $u' \in L^p(0,T;V)$, $u'' \in L^q(0,T;V^\star)$ and (14.1), (14.2) hold.*

The proof of this Theorem follows from the proof of Theorem 14.1.

**Remark 14.3.** One can prove the following generalization of Theorem 14.1 to problems with nonhomogeneous initial conditions. Assume $(D1)$ - $(D3)$ or $(D1')$, $(D2)$, $(D3')$ such that the coercivity of $N$ holds in the sense of Theorem 9.6. Then for arbitrary $F \in L^q(0,T;V^\star)$, $u_0 \in V$, $u_1 \in H$ there exists $u$ such that $u \in C([0,T];V)$, $u' \in L^p(0,T;V)$, $u'' \in L^q(0,T;V^\star)$, $u$ satisfies (14.1) and

$$u(0) = u_0, \quad u'(0) = u_1. \tag{14.24}$$

Indeed, if $u$ is a solution of (14.1), (14.24) then $\tilde v = u' \in L^p(0,T;V)$, $\tilde v' \in L^q(0,T;V^\star)$ and $\tilde v$ satisfies

$$\tilde v' + N(\tilde v) + QS\tilde v + M(\tilde v, S\tilde v + u_0) = F - \tilde Q u_0, \tag{14.25}$$

$$\tilde v(0) = u_1. \tag{14.26}$$

Conversely, if $\tilde{v} \in L^p(0,T;V)$ satisfies (14.25), (14.26) then $u = S\tilde{v} + u_0$ satisfies (14.1), (14.24), $u \in C([0,T];V)$.

It is not difficult to show that if $M$ satisfies $(D3)$ or $(D3')$ then the operator

$$(\tilde{v}, \tilde{u}) \mapsto M(\tilde{v}, \tilde{u} + u_0)$$

also satisfies (iii) or (iii'). Consequently, by Theorems 9.5, 14.1, 14.2 there is a solution of (14.25), (14.26) and so there is a solution of (14.1), (14.24).

**Remark 14.4.** Assume that $(D1)$ is satisfied such that $N$ is uniformly monotone in the sense

$$\langle [N(v)](t) - [N(w)](t), v(t) - w(t)\rangle \geq c_2 \|v(t) - w(t)\|_V^p \text{ and } \tag{14.27}$$

$$\langle \tilde{Q}(\tilde{u}), \tilde{u}\rangle \geq c_3 \|\tilde{u}\|_V^2$$

with some positive constants $c_2, c_3$, further, $M = 0$. Then the solution of (14.1), (14.24) is unique and it depends continuously on $F, u_0, u_1$.

Indeed, then for solutions $\tilde{v}_j$ of (14.25), (14.26) with $f = F_j$, $u_0 = u_0^j$, $u_1 = u_1^j$ $(j = 1, 2)$ we have

$$\int_0^t \langle \tilde{v}_1'(\tau) - \tilde{v}_2'(\tau), \tilde{v}_1(\tau) - \tilde{v}_2(\tau)\rangle d\tau + \tag{14.28}$$

$$\int_0^t \langle [N(\tilde{v}_1)](\tau) - [N(\tilde{v}_2)](\tau), \tilde{v}_1(\tau) - \tilde{v}_2(\tau)\rangle d\tau +$$

$$\int_0^t \langle QS(\tilde{v}_1(\tau) - \tilde{v}_2(\tau)), \tilde{v}_1(\tau) - \tilde{v}_2(\tau)\rangle d\tau =$$

$$\int_0^t \langle \tilde{F}_1(\tau) - \tilde{F}_2(\tau), \tilde{v}_1(\tau) - \tilde{v}_2(\tau)\rangle d\tau - \int_0^t \langle \tilde{Q}(u_0^1 - u_0^2), \tilde{v}_1(\tau) - \tilde{v}_2(\tau)\rangle d\tau.$$

Since by Remark 6.7

$$\int_0^t \langle \tilde{v}_1'(\tau) - \tilde{v}_2'(\tau), \tilde{v}_1(\tau) - \tilde{v}_2(\tau)\rangle d\tau = \frac{1}{2}\|\tilde{v}_1(t) - \tilde{v}_2(t)\|_H^2 - \frac{1}{2}\|\tilde{v}_1(0) - \tilde{v}_2(0)\|_H^2,$$

we obtain from (14.27), (14.28) by Young's inequality for the solutions $\tilde{u}_j = S\tilde{v}_j + u_0^j$

$$\frac{1}{2}\|\tilde{u}_1'(t) - \tilde{u}_2'(t)\|_H^2 + \frac{c_2}{2}\|\tilde{u}_1' - \tilde{u}_2'\|_{L^p(0,t;V)}^p + \frac{c_3}{2}\|\tilde{u}_1(t) - \tilde{u}_2(t)\|_V^2 \leq$$

$$c_4\|F_1 - F_2\|_{L^q(0,t;V^\star)}^q + c_5\|u_1^1 - u_1^2\|_H^2 + c_6\|u_0^1 - u_0^2\|_V^2$$

with some positive constants $c_4, c_5, c_6$.

Applying Theorems 10.1, 10.9, one easily gets from Theorem 14.1 and Remark 14.3

**Theorem 14.5.** *Let $V$ be a closed linear subspace of $W^{1,p}(\Omega)$, ($p \geq 2$, $\Omega \subset \mathbb{R}^n$ a bounded domain with sufficiently smooth boundary), $H = L^2(\Omega)$. Assume that $N : L^p(0,T;V) \to L^q(0,T;V^\star)$ has the form (10.37) and (C1) – (C5) are fulfilled such that $g_2(u)$ and $k_2(u)$ are not depending on $u$. Further, operator $\tilde{Q}$ has the form*

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \int_\Omega \left[\sum_{j,l=1}^n a_{jl}(x)(D_l\tilde{u})(D_j\tilde{v}) + d(x)\tilde{u}\tilde{v}\right] dx + \tag{14.29}$$

$$\int_{\Omega\times\Omega} K(x,z)\tilde{u}(x)\tilde{v}(z)dxdz, \quad \tilde{u},\tilde{v} \in V,$$

*where $a_{jl}, d \in L^\infty(\Omega)$, $a_{jl} = a_{lj}$, $\sum_{j,l=1}^n a_{jl}(x)\xi_j\xi_l \geq 0$, $d(x) \geq 0$ for a.a. $x \in \Omega$, all $\xi \in \mathbb{R}^n$,*

$$K \in L^2(\Omega\times\Omega), \quad K(x,z) = K(z,x) \text{ and}$$

$$\int_{\Omega\times\Omega} K(x,z)\tilde{u}(x)\tilde{u}(z)dxdz \geq 0. \tag{14.30}$$

*Finally, there is $\delta > 0$ with $\delta < 1/p$ such that*

$$M : L^p(0,T;V) \times L^p(0,T;V) \to L^q(0,T;W^{1-\delta,p}(\Omega)^\star) \tag{14.31}$$

*is bounded, demicontinuous,*

$$\|M(v,u)\|_{L^q(0,T;W^{1-\delta,p}(\Omega)^\star)} \leq const\left[\|v\|^\sigma_{L^p(0,T;V)} + \|u\|^\sigma_{L^p(0,T;V)}\right] \tag{14.32}$$

*with some constant $0 < \sigma < p-1$.*

*Then there exists a solution of (14.1), (14.24).*

*Proof.* By Theorem 10.9, and Remark 10.10 $N$ satisfies $(D1')$. Clearly, $\tilde{Q}$ satisfies $(D2)$. Finally, we show that $M$ satisfies $(D3')$. By (14.31)

$$M : L^p(0,T;V) \times L^p(0,T;V) \to L^q(0,T;V^\star)$$

is bounded and demicontinuous. Further, if

$$(u_k) \to u, \quad (u_k') \to u' \text{ weakly in } L^p(0,T;V),$$

$$(u_k'') \to u'' \text{ weakly in } L^q(0,T;V^\star)$$

then by Theorem 10.1 for a subsequence

$$(u_k') \to u' \text{ in } L^p(0,T;W^{1-\delta,p}(\Omega)),$$

thus by Hölder's inequality

$$[M(u_k', u_k), u_k' - u'] \to 0$$

since $M(u_k', u_k)$ is bounded in $L^q(0,T;W^{1-\delta,p}(\Omega)^\star)$ by (14.31). The assumption (14.32) implies (14.23). Therefore, from Theorem 14.2 we obtain Theorem 14.5. □

**Remark 14.6.** The assumption (14.30) means that the selfadjoint and compact operator $\tilde{K} : L^2(\Omega) \to L^2(\Omega)$, defined by

$$(\tilde{K}\tilde{v})(x) = \int_\Omega K(x,z)\tilde{v}(z)dz, \quad \tilde{v} \in L^2(\Omega)$$

is positive which is equivalent to the fact that all eigenvalues of $\tilde{K}$ are nonnegative which holds if and only if the function $K$ has the form

$$K(x,z) = \sum_j \psi_j(x)\psi_j(z) \text{ with some } \psi_j \in L^2(\Omega). \tag{14.33}$$

Indeed, by the Hilbert–Schmidt theorem

$$\tilde{K}\tilde{v} = \sum_j \lambda_j(\tilde{v}, \varphi_j)\varphi_j$$

where $\lambda_j$ are the eigenvalues and $\varphi_j, j = 1, 2, ...$ is the orthonormal system of the corresponding eigenfunctions of $\tilde{K}$ (this system is finite or countably infinite). Thus

$$\int_{\Omega\times\Omega} K(x,z)\tilde{u}(x)\tilde{u}(z)dxdz = (\tilde{K}\tilde{u}, \tilde{u}) = \sum_j \lambda_j |(\tilde{u}, \varphi_j)|^2.$$

Further, since

$$\int_{\Omega\times\Omega} K(x,z)\tilde{u}(x)\tilde{v}(z)dxdz = \int_\Omega (\tilde{K}\tilde{v})(x)\tilde{u}(x)dx =$$

$$\int_\Omega \tilde{u}(x)\left[\sum_j \lambda_j(\tilde{v}, \varphi_j)\varphi_j(x)\right]dx = \int_{\Omega\times\Omega}\left[\sum_j \lambda_j\varphi_j(x)\varphi_j(z)\right]\tilde{u}(x)\tilde{v}(z)dxdz,$$

and $\lambda_j \geq 0$, we have

$$K(x,z) = \sum_j \lambda_j\varphi_j(x)\varphi_j(z),$$

i.e. we have (14.33) with $\psi_j = \lambda_j^{1/2}\varphi_j$.

## Problems

1. Prove Theorem 14.2.

2. Consider the initial-boundary value problem

$$D_t^2 u - \sum_{j=1}^n D_j[a_j(t,x,D_tu,DD_tu)] + a_0(t,x,D_tu,DD_tu) - \tag{14.34}$$

$$\sum_{j,l=1}^n D_j[a_{jl}(x)D_lu] + d(x)u = f(t,x), \quad (t,x) \in Q_T,$$

$$u(0,x) = u_0(x), \quad D_t u(0,x) = u_1(x), \quad x in \Omega, \tag{14.35}$$

$$u|_{\Gamma_T} = 0 \text{ where } \Gamma_T = [0,T) \times \partial\Omega. \tag{14.36}$$

Prove that $u$ is a ("sufficiently smooth") classical solution of (14.34) – (14.36) if and only if the function $U$, defined by $U(t) = x \mapsto u(t,x)$ satisfies (14.1), (14.24) where $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$, $M = 0$,

$$[N(v), w] = \int_{Q_T} \left[ \sum_{j=1}^n a_j(t,x,v,Dv) D_j w + a_0(t,x,v,Dv) w \right] dtdx, \tag{14.37}$$

$$[Qu, w] = \int_{Q_T} \left[ \sum_{j,l=1}^n a_{jl}(x)(D_l u) D_j w + d(x) uw \right] dtdx, \tag{14.38}$$

$$u, v, w \in L^p(0,T;V),$$

$$[F, w] = \int_{Q_T} fw dtdx, \quad w \in L^p(0,T;V). \tag{14.39}$$

If this function $U$ satisfies (14.1), (14.24) with the operators (14.37), (14.38) and with $F$ defined in (14.39), it is called a weak solution of (14.34) – (14.36).

3. Assume that $M = 0$,

$$[N(v), w] = \int_{Q_T} \left[ \sum_{j=1}^n f_j(t,x,D_j v) D_j w + f_0(t,x,D_j v) w \right] dtdx$$

where the functions $f_j$ satisfy the Carathéodory conditions,

$$\xi_j \mapsto f_j(t,x,\xi_j) \text{ is monotone nondecreasing}$$

for a.a. $(t,x) \in Q_T$, $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$,

$$\beta_1 |\xi_j|^{p-1} \le |f_j(t,x,\xi_j)| \le \beta_2 |\xi_j|^{p-1} \text{ for a.a. } (t,x) \in Q_T$$

with some positive constants $\beta_1, \beta_2$ and $p \ge 2$. Further, $Q$ has the form (14.38) where

$$a_{jl}, d \in L^\infty(\Omega), \quad a_{jl} = a_{lj}, \quad \sum_{j,l=1}^n a_{jl}(x)\xi_j\xi_l \ge 0, \quad d(x) \ge 0 \tag{14.40}$$

for a.a. $x \in \Omega$, all $\xi \in \mathbb{R}^n$.

Prove that then for each $F \in L^q(0,T;V^\star)$, $u_0 \in V$, $u_1 \in H$ there exists a solution of (14.1), (14.24) (i.e. a weak solution of (14.34) – (14.36) with $a_j(t,x,\xi) = f_j(t,x,\xi_j)$, $j = 0.1, \ldots, n$).

4. Let $M = 0$ and

$$[N(v), w] = \int_{Q_T} \left[ \sum_{j=1}^{n} (D_j v)|Dv|^{p-2} D_j w + v|v|^{p-2} w \right] dt dx,$$

where $v, w \in L^p(0,T;V)$, $V = W_0^{1,p}(\Omega)$, $p \geq 2$, $H = L^2(\Omega)$. Further, assume that $Q$ has the form (14.38) such that conditions (14.40) hold.

Prove that then for each $F \in L^q(0,T;V^\star)$, $u_0 \in V$, $u_1 \in H$ there exists a solution of (14.1), (14.24), i.e. a weak solution of (14.34) – (14.36) with

$$a_j(t,x,\eta,\zeta) = \zeta_j|\zeta|^{p-2} \text{ for } j = 1,\ldots,n,$$

$$a_0(t,x,\eta,\zeta) = \eta|\eta|^{p-2}, \quad \eta \in \mathbb{R}, \quad \zeta \in \mathbb{R}^n.$$

# 15 Solutions in $(0,\infty)$

Now we consider equation (14.1) for $t \in (0,\infty)$. By using the notations of Section 11 we have

**Theorem 15.1.** *Assume that $\tilde{Q} : V \to V^\star$ satisfies (ii). Let*

$$N : L^p_{loc}(0,\infty;V) \to L^q_{loc}(0,\infty;V^\star),$$

$$M : L^p_{loc}(0,\infty;V) \times L^p_{loc}(0,\infty;V) \to L^q_{loc}(0,\infty;V^\star)$$

*be operators of Volterra type and assume that for each finite $T > 0$ their restrictions to $(0,T)$ satisfy (D1) and (D3) such that the coercivity of $N$ holds in the sense of Theorem 9.6.*

*Then for arbitrary $F \in L^q(0,\infty;V^\star)$, $u_0 \in V$, $u_1 \in H$ there exists $u$ such that $u \in C([0,\infty);V)$, $u' \in L^p_{loc}(0,\infty;V)$, $u'' \in L^q_{loc}(0,\infty;V^\star)$ and*

$$u''(t) + [N(u')](t) + \tilde{Q}u(t) + [M(u',u)](t) = F(t) \text{ for a.a. } t \in (0,\infty), \tag{15.1}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{15.2}$$

The proof is similar to that of Theorem 11.4, based on Remark 14.3.

From Theorems 14.5, 11.9 we obtain

**Theorem 15.2.** *Let $V$ be a closed linear subspace of $W^{1,p}(\Omega)$, $2 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ a bounded domain with sufficiently smooth boundary. Assume that the functions*

$$a_j : Q_\infty \times \mathbb{R}^{n+1} \times L^p_{loc}(0,\infty;V) \to \mathbb{R}, \quad j = 0, 1, ..., n$$

*satisfy the assumptions of Theorem 11.9, $N$ has the form (11.14), $\tilde{Q} : V \to V^\star$ satisfies the assumptions of Theorem 14.5,*

$$M : L^p_{loc}(0,\infty;V) \times L^p_{loc}(0,\infty;V) \to L^q_{loc}(0,\infty;W^{1-\delta,p}(\Omega)^\star)$$

*is of Volterra type and satisfies the assumptions of Theorem 14.5 for arbitrary finite* $T > 0$.

*Then for arbitrary* $F \in L^q_{loc}(0,\infty;V^\star)$, $u_0 \in V$, $u_1 \in L^2(\Omega)$ *there exists* $u \in L^p_{loc}(0,\infty;V)$ *such that* $u' \in L^p_{loc}(0,\infty;V)$, $u'' \in L^q_{loc}(0,\infty;V^\star)$,

$$u'' + N(u') + Qu + M(u',u) = F \text{ in } (0,\infty), \quad u(0) = u_0, \quad u'(0) = u_1.$$

Now we formulate a theorem on boundedness of the solutions $u$ of (15.1), (15.2).

**Theorem 15.3.** *Let the assumptions of Theorem 15.1 be satisfied such that with some* $c_2 > 0$

$$\langle [N(v)](t), v(t)\rangle \geq c_2 \|v(t)\|_V^p, \quad t \in (0,\infty) \tag{15.3}$$

*for all* $u, v \in L^p_{loc}(0,\infty;V)$, *and with some nonnegative* $\Phi_1, \Phi_2 \in L^1(0,\infty)$, *a positive constant* $\tilde\sigma < 1$, $y(\tau) = \|v(\tau)\|_H^2$ *we have*

$$\|[M(v,u)](t)\|_{V^\star}^q \leq \Phi_1(t) \sup_{[0,t]} y^{\tilde\sigma} + \Phi_2(t), \quad t \in (0,\infty) \tag{15.4}$$

*Finally, let* $F \in L^q(0,\infty;V^\star)$.

*Then for a solution* $u$ *of (15.1), (15.2),* $y(t) = \|u'(t)\|_H^2$ *is bounded in* $(0,\infty)$, $u' \in L^p(0,\infty;V)$ *and*

$$\langle \tilde Q[u(t)], u(t)\rangle \text{ is bounded for } t \in (0,\infty). \tag{15.5}$$

*If*

$$\langle \tilde Q \tilde u, \tilde u\rangle \geq c_3 \|\tilde u\|^2_{W^{1,2}(\Omega)} \text{ for all } \tilde u \in V \tag{15.6}$$

*with some constant* $c_3 > 0$ *then*

$$\|u(t)\|_{W^{1,2}(\Omega)} \text{ is bounded for } t \in (0,\infty). \tag{15.7}$$

*Proof.* Applying both sides of (15.1) to $u'$ and integrating over $[0,T]$, we obtain

$$[u'', u'] + [N(u'), u'] + [Qu, u'] + [M(u',u), u'] = [F, u']. \tag{15.8}$$

By Remark 6.7 and (6.9)

$$[u'', u'] = \frac{1}{2}\|u'(T)\|_H^2 - \frac{1}{2}\|u'(0)\|_H^2 = \frac{1}{2} y(T) - \frac{1}{2} y(0) \tag{15.9}$$

and by (14.8)

$$[Qu, u'] = \frac{1}{2}\langle \tilde Q u(T), u(T)\rangle - \frac{1}{2}\langle \tilde Q u(0), u(0)\rangle. \tag{15.10}$$

Further, by Young's inequality

$$|[M(u', u), u']| \leq \frac{\varepsilon^p}{p} \int_0^T \|u'(t)\|_V^p dt + \frac{1}{\varepsilon^q q} \int_0^T \|[M(u', u)](t)\|_{V^\star}^q dt, \quad (15.11)$$

$$|[F, u']| \leq \frac{\varepsilon^p}{p} \int_0^T \|u'(t)\|_V^p dt + \frac{1}{\varepsilon^q q} \int_0^T \|F(t)\|_{V^\star}^q dt. \quad (15.12)$$

Choosing sufficiently small $\varepsilon > 0$, from (15.3), (15.4), (15.8) – (15.12) we obtain the inequality

$$\frac{1}{2} y(T) + \frac{c_2}{2} \int_0^T \|u'(t)\|_V^p dt + \frac{1}{2} \langle \tilde{Q} u(T), u(T) \rangle \leq \quad (15.13)$$

$$\text{const} \left[ \int_0^T \|[M(u', u)](t)\|_{V^\star}^q dt + \int_0^T \|F(t)\|_{V^\star}^q dt \right] + \frac{1}{2} y(0) +$$

$$\frac{1}{2} \langle \tilde{Q} u(0), u(0) \rangle \leq \text{const} \left[ 1 + \sup_{[0,T]} y^{\tilde{\sigma}} \int_0^T \Phi_1(t) dt + \int_0^T \Phi_2(t) + \int_0^T \|F(t)\|_{V^\star}^q dt \right].$$

Since $\tilde{\sigma} < 1$, $\Phi_1, \Phi_2 \in L^1(0, \infty)$, $F \in L^q(0, \infty; V^\star)$, we obtain from (15.13) that $y(T)$ and $\langle \tilde{Q} u(T), u(T) \rangle$ are bounded for $T \in (0, \infty)$ and $u' \in L^p(0, \infty; V)$. Finally, (15.6) implies (15.7). □

Now we consider examples for operators $N, M, \tilde{Q}$ which satisfy the assumptions of Theorems 14.5 - 15.3.

The operator in Example 10.12 satisfies the conditions on $N$ in Theorem 14.5 and the operator in Example, considered in Section 12 satisfies the conditions on $N$ in Theorem 15.2. In the case $b, b_0 \geq c_2$ with some positive constant $c_2$ and $\hat{b}_0 = 0$ the assumption on $N$ in Theorem 15.3 are fulfilled.

It is easy to show that the assumptions on $M$ in Theorem 14.5 are fulfilled if e.g.

$$[M(v, u)](w) = \int_{Q_T} g(t, x, [G_1(v)](t), [G_2(v)](t)) w \, dt dx + \int_{\Gamma_T} h_2(t, x; u) w \, d\sigma \quad (15.14)$$

$$u, v \in L^p(0, T; V), \quad w \in L^p(0, T; W^{1-\delta, p}(\Omega))$$

where $g$ is a Carathéodory function satisfying with some positive constant $\sigma < p - 1$

$$|g(t, x, \theta_1, \theta_2)| \leq \text{const}[1 + |\theta_1|^\sigma + |\theta_2|^\sigma], \quad (15.15)$$

$G_1, G_2 : L^p(0, T; V) \to L^p(Q_T)$ are linear and continuous operators, $0 < \delta < 1/p$, $\Gamma_T = (0, T) \times \partial\Omega$,

$$h_2 : \Gamma_T \times L^p(0, T; V) \to \mathbb{R}$$

is a measurable function, satisfying

$$\|h_2(t, x; u)\|_{L^q(\Gamma_T)} \leq \text{const} \left[ 1 + \|u\|_{L^p(0,T;V)} \right]^\sigma.$$

Further, the assumptions on $M$ in Theorem 15.2 are satisfied if (15.14), (15.15) hold for all $t \in (0, \infty)$,

$$G_1, G_2 : L^p_{loc}(0, \infty; V) \to L^p_{loc}(Q_\infty)$$

are linear operators of Volterra type and for all fixed finite $T > 0$, they map $L^p(0, T; V)$ into $L^p(Q_T)$ continuously.

The assumptions on $M$ in Theorem 15.3 are satisfied if

$$|g(t, x, \theta_1, \theta_2)| \leq \Phi_1(t)|\theta_1|^\lambda + \Phi_2(t), \quad t \in (0, \infty)$$

with $\Phi_1, \Phi_2 \in L^q(0, \infty) \cap L^\infty(0, \infty)$, $0 \leq \lambda < 2/q$ and for all $v \in L^p_{loc}(Q_\infty)$

$$\|G_1(v)\|_{L^p(Q_t)} \leq \text{const}\|v\|_{L^2(Q_t)}, \quad t \in (0, \infty).$$

Finally, (15.6) is satisfied for the operator $\tilde{Q}$ of the form (14.29) if for a.a. $x \in \Omega$, all $\xi = (\xi_0, \xi_1, ..., \xi_n) \in \mathbb{R}^{n+1}$

$$\sum_{j,l=1}^{n} a_{jl}(x)\xi_j\xi_l + d(x)\xi_0^2 \geq c_3|\xi|^2 \tag{15.16}$$

with some constant $c_3 > 0$.

Now we shall formulate conditions which imply a result on the stabilization of solutions $u$ of (15.1) as $t \to \infty$. For simplicity we consider the case when $N$ is "local", i.e. $[N(u)](t) = [\tilde{N}(t)](u(t))$ where $\tilde{N}(t) : V \to V^\star$ is defined for all $t > 0$ and $M = 0$.

**Theorem 15.4.** *Assume that the operator $N : L^p_{loc}(0, \infty; V) \to L^q_{loc}(0, \infty; V^\star)$ is given by $[N(u)](t) = [\tilde{N}(t)](u(t))$ where $\tilde{N}(t) : V \to V^\star$ satisfies the assumptions of Theorem 7.1 such that for all $\tilde{v} \in V$*

$$\langle [\tilde{N}(t)](\tilde{v}), \tilde{v}\rangle \geq c_2(t+1)^\mu \|\tilde{v}\|^p_V \tag{15.17}$$

*with some constants $\mu > p - 1$ ($p \geq 2$), $c_2 > 0$. (In this case $\tilde{N}(t)$ is uniformly monotone, see Definition 2.15.) The operator $M = 0$ and $\tilde{Q}$ satisfies (D2) and (15.6). Further, there exist $F_\infty \in V^\star$, a continuous function $\Phi \geq 0$ with*

$$\lim_\infty \Phi = 0, \quad \int_0^\infty \Phi(t)^q dt < \infty \tag{15.18}$$

*such that*

$$\|F(t) - F_\infty\|_{V^\star} \leq \Phi(t) \tag{15.19}$$

*and there exists a solution $u_\infty \in V$ of*

$$\tilde{Q}u_\infty = F_\infty \tag{15.20}$$

*Then for a solution $u$ of (15.1) with $M = 0$ we have*

$$\lim_{t\to\infty} \|u'(t)\|_H = 0, \tag{15.21}$$

$$\int_0^\infty (t+1)^\beta \|u'(t)\|_H^2 dt < \infty, \quad \int_0^\infty (t+1)^\mu \|u'(t)\|_V^p dt < \infty \tag{15.22}$$

*where* $0 \le \beta < [2\mu - (p-2)]/p$ *and there exists* $w \in V$ *such that*

$$\|u(t) - w\|_V^q \le \frac{const}{\lambda - 1} \frac{1}{(t+1)^{\lambda - 1}} \tag{15.23}$$

*where* $\lambda = \mu/(p-1) > 1$.

*Proof.* Since $u_\infty \in V$ and so its derivative with respect to $t$ is 0, we may apply (15.1) to $u' = (u - u_\infty)'$, and thus, integrating over $[0, T]$ we obtain by (15.20)

$$\int_0^T \langle u''(t), u'(t)\rangle dt + \int_0^T \langle [N(u')](t), u'(t)\rangle dt + \tag{15.24}$$

$$\int_0^T \langle \tilde{Q}[u(t) - u_\infty], [u(t) - u_\infty]'\rangle dt = \int_0^T \langle F(t) - F_\infty, u'(t)\rangle dt.$$

By using the notation $y(t) = \|u'(t)\|_H^2$, we obtain by Remark 6.7 and (6.9)

$$\int_0^T \langle u''(t), u'(t)\rangle dt = \frac{1}{2} y(T) - \frac{1}{2} y(0) \tag{15.25}$$

(see (15.9)) and by (14.8)

$$\int_0^T \langle \tilde{Q}[u(t) - u_\infty], [u(t) - u_\infty]'\rangle dt = \tag{15.26}$$

$$\frac{1}{2}\langle \tilde{Q}[u(T) - u_\infty], u(T) - u_\infty\rangle - \frac{1}{2}\langle \tilde{Q}[u(0) - u_\infty], u(0) - u_\infty\rangle.$$

Further, by Young's inequality

$$\left| \int_0^T \langle F(t) - F_\infty, u'(t)\rangle dt \right| \le \tag{15.27}$$

$$\frac{\varepsilon^p}{p} \int_0^T \|u'(t)\|_V^p dt + \frac{1}{\varepsilon^q q} \int_0^T \|F(t) - F_\infty\|_{V^\star}^q dt.$$

Choosing sufficiently small $\varepsilon > 0$, by (15.17), (15.19), (15.24) - (15.27) we find

$$\frac{1}{2} y(T) + \frac{c_2}{2} \int_0^T (t+1)^\mu \|u'(t)\|_V^p dt + \frac{1}{2}\langle \tilde{Q}[u(T) - u_\infty], u(T) - u_\infty\rangle \le \tag{15.28}$$

$$\text{const} \int_0^T [\Phi(t)]^q dt + \frac{1}{2} y(0) + \frac{1}{2}\langle \tilde{Q}[u(0) - u_\infty], u(0) - u_\infty\rangle.$$

Since the right hand side is bounded for all $T > 0$ by (15.18), we obtain the second part of (15.22), i.e.

$$\int_0^\infty (t+1)^\mu \|u'(t)\|_V^p dt < \infty. \tag{15.29}$$

Consequently, for any $T_1 < T_2$ we have

$$\|u(T_2) - u(T_1)\|_V = \|(Su')(T_2) - (Su')(T_1)\|_V = \| \int_{T_1}^{T_2} u'(t)dt\|_V \leq \tag{15.30}$$

$$\int_{T_1}^{T_2} \|u'(t)\|_V dt = \int_{T_1}^{T_2} \frac{1}{(t+1)^{\lambda/q}}(t+1)^{\lambda/q}\|u'(t)\|_V dt \leq$$

$$\left\{\int_{T_1}^{T_2} \frac{1}{(t+1)^{\lambda}}dt\right\}^{1/q} \left\{\int_{T_1}^{T_2} (t+1)^{\mu}\|u'(t)\|_V^p dt\right\}^{1/p}$$

where $\lambda = \mu/(p-1) > 1$ and thus $p\lambda/q = \lambda(p-1) = \mu$.

Thus, for any $\varepsilon > 0$ there exists $T_0$ such that for $T_0 < T_1 < T_2$

$$\|u(T_2) - u(T_1)\|_V < \varepsilon.$$

Hence, there exists $w \in V$ such that

$$\lim_{T\to\infty} \|u(T) - w\|_V = 0. \tag{15.31}$$

In order to prove (15.23), take the limit $T_2 \to +\infty$ in (15.30), then we find

$$\|w - u(T_1)\|_V \leq \int_{T_1}^{\infty} \|u'(t)\|_V dt \leq$$

$$\left\{\int_{T_1}^{\infty} \frac{1}{(t+1)^{\lambda}}dt\right\}^{1/q} \left\{\int_{T_1}^{\infty} (t+1)^{\mu}\|u'(t)\|_V^p dt\right\} \leq$$

$$\left\{\frac{1}{\lambda-1}\frac{1}{(T_1+1)^{\lambda-1}}\right\}^{1/q} \int_0^{\infty} (t+1)^{\mu}\|u'(t)\|_V^p dt,$$

i.e. we have (15.23).

The first estimate in (15.22) can be obtained as follows. If $0 \leq \beta < [2\mu - (p-2)]/p$ then by Hölder's inequality

$$\int_0^{\infty} (t+1)^{\beta}\|u'(t)\|_H^2 dt \leq \text{const} \int_0^{\infty} (t+1)^{\beta}\|u'(t)\|_V^2 dt =$$

$$\text{const} \int_0^{\infty} (t+1)^{\beta-2\mu/p} \left[(t+1)^{2\mu/p}\|u'(t)\|_V^2\right] dt \leq$$

$$\text{const}\left\{\int_0^{\infty} (t+1)^{\frac{\beta p-2\mu}{p-2}} dt\right\}^{(p-2)/p} \left\{\int_0^{\infty} (t+1)^{\mu}\|u'(t)\|_V^p dt\right\}^{2/p} < \infty$$

because of the second part of (15.22) and $\frac{\beta p-2\mu}{p-2} < -1$. In the case $p = 2$ the first multiplier in the last term is the $L^{\infty}(0,\infty)$ norm of the function $t \mapsto (t+1)^{\beta-2\mu/p}$.

Now we apply again (15.1) to $u' = (u - u_\infty)'$ and integrate over $[T_1, T_2]$, then we obtain by (15.20)

$$\int_{T_1}^{T_2} \langle u''(t), u'(t)\rangle dt + \int_{T_1}^{T_2} \langle [N(u')](t), u'(t)\rangle dt +$$

$$\int_{T_1}^{T_2} \langle \tilde{Q}[u(t) - u_\infty], [u(t) - u_\infty]'\rangle dt = \int_{T_1}^{T_2} \langle F(t) - F_\infty, u'(t)\rangle dt,$$

whence, similarly to (15.28), we find

$$\frac{1}{2}[y(T_2) - y(T_1)] + \frac{c_2}{2}\int_{T_1}^{T_2} (t+1)^\mu \|u'(t)\|_V^p dt + \tag{15.32}$$

$$\frac{1}{2}\langle \tilde{Q}[u(T_2) - u_\infty], u(T_2) - u_\infty\rangle - \frac{1}{2}\langle \tilde{Q}[u(T_1) - u_\infty], u(T_1) - u_\infty\rangle \le \text{const} \int_{T_1}^{T_2} [\Phi(t)]^q dt.$$

Since $\tilde{Q} : V \to V^\star$ is a continuous and linear operator, by (15.31)

$$\lim_{T_1, T_2 \to \infty} \{\langle \tilde{Q}[u(T_2) - u_\infty], u(T_2) - u_\infty\rangle - \langle \tilde{Q}[u(T_1) - u_\infty], u(T_1) - u_\infty\rangle\} = 0,$$

thus (15.18), (15.29) imply

$$\lim_{T_1, T_2 \to \infty} [y(T_1) - y(T_2)] = 0.$$

Thus $\lim_{T\to\infty} y(T)$ exists and is finite, further, by the first estimate in (15.22), it must be 0, i.e. we have (15.21) which completes the proof of Theorem 15.4 $\square$

The following example satisfies the assumptions of Theorem 15.4.

**Example 15.5.** Set $[N(u)](t) = [\tilde{N}(t)][u(t)]$ where

$$\langle [\tilde{N}(t)](\tilde{u}), \tilde{v}\rangle = (t+1)^\mu \int_\Omega \left[ |\nabla \tilde{u}|^{p-2} \sum_{j=1}^n (D_j\tilde{u})(D_j\tilde{v}) + |\tilde{u}|^{p-2}\tilde{u}\tilde{v} \right],$$

$\tilde{u}, \tilde{v} \in V$ where $V = W_0^{1,p}(\Omega)$ or $V = W^{1,p}(\Omega)$, $\mu > p-1$, $p \ge 2$, $M = 0$ and

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \int_\Omega \left[ \sum_{j,l=1}^n a_{jl}(D_j\tilde{u})(D_l\tilde{v}) + d\tilde{u}\tilde{v} \right], \quad \tilde{u}, \tilde{v} \in V$$

where the functions $a_{jl}, d \in C(\overline{\Omega})$ satisfy the (uniform ellipticity) condition (15.16). Finally,

$$F(t) = F_\infty + \Phi(t)g \text{ where } F_\infty \in L^p(\Omega), \quad g \in L^q(\Omega).$$

It is well-known (see, e.g. [2]) that for a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary and $F_\infty \in L^p(\Omega)$, there exists a unique solution $\tilde{u} \in W^{2,p}(\Omega)$ solution of the linear equation

$$-\sum_{j,l=1}^{n} D_l(a_{jl} D_j \tilde{u}) + d\tilde{u} = F_\infty \text{ in } \Omega$$

with the boundary condition

$$\tilde{u} \mid_{\partial\Omega} = 0, \quad (\partial_\nu^A \tilde{u}) \mid_{\partial\Omega} = 0,$$

respectively, where $\partial_\nu^A$ denotes the "conormal derivative" of $\tilde{u}$ on $\partial\Omega$ (with respect to the differential operator in the differential equation). Thus we have a solution of (15.20).

## Problems

1. Show that the operator $M$ defined by (15.14) satisfies the assumptions of Theorem 14.5.
2. Show that the Example 15.5 satisfies the assumptions of Theorem 15.4.
3. Formulate and prove an existence theorem on problem (15.1), (15.2) with the operators $M = 0$, $N, Q$ considered in Problem 3 of Section 14 with arbitrary $t \in (0, \infty)$.
4. Formulate and prove an existence theorem on problem (15.1), (15.2) with the operators $M = 0$, $N, Q$ considered in Problem 4 of Section 14 with arbitrary $t \in (0, \infty)$.

# 16 Semilinear hyperbolic equations

In this section we shall consider the equation (14.1) in the case when $N = 0$ and operator $M$ has a particular form (see (16.2)), further, $V$ is a closed linear subspace of $W^{1,2}(\Omega)$, $(p = 2)$, $H = L^2(\Omega)$.

## Existence of solutions in $[0, T]$

**Theorem 16.1.** *Let $V \subset W^{1,2}(\Omega)$ be a closed linear subspace, $p = 2$, $H = L^2(\Omega)$. Assume that $\tilde{Q} : V \to V^\star$ satisfies (D2) (see Section 14) and*

$$\langle \tilde{Q}\tilde{u}, \tilde{u} \rangle \geq c_0 \|\tilde{u}\|^2_{W^{1,2}(\Omega)} \text{ for all } \tilde{u} \in V \tag{16.1}$$

*with some constant $c_0 > 0$ (i.e. $\tilde{Q}$ satisfies (15.6)).*

*Let operator $M(u, u')$ have the form*

$$[M(u, u')](t, x) = \varphi(x) h'(u(t)) + \psi(t, x; u) u'(t) \text{ where} \tag{16.2}$$

$$\varphi : \Omega \to \mathbb{R}, \quad \psi : Q_T \times L^2(Q_T) \to \mathbb{R}$$

*are measurable in $x$ and $(t,x)$, respectively, $\psi$ has the Volterra property and*

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ implies } \psi(t,x;u_k) \to \psi(t,x;u) \text{ for a.a. } (t,x) \in Q_T,$$

*for a subsequence. Further, there exist positive constants $c_1, c_2, c_3$ such that*

$$c_1 \leq \varphi(x) \leq c_2, \quad 0 \leq \psi(t,x;u) \leq c_3; \tag{16.3}$$

*$h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function satisfying*

$$h(\eta) \geq 0, \quad |h'(\eta)| \leq const\, |\eta|^{\varrho+1} \text{ where } 0 \leq \varrho \leq \frac{2}{n-2}. \tag{16.4}$$

*(In the case $n = 2$, $\varrho + 1$ may be any nonnegative number.)*

*Then for any $F \in L^2(0,T;H)$, $u_0 \in V$, $u_1 \in H$ there exists $u \in L^\infty(0,T;V)$ such that*

$$u' \in L^\infty(0,T;H), \quad u'' \in L^2(0,T;V^\star), \tag{16.5}$$

$$u''(t) + (Qu)(t) + \varphi(x)h'(u(t)) + \psi(t,x;u)u'(t) = F(t) \text{ for a.a. } t \in [0,T], \tag{16.6}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{16.7}$$

**Remark 16.2.** One can show (see, e.g [93]) that $H$ is dense in the Hilbert space $V^\star$, thus

$$H \subset V^\star \subset H^\star \tag{16.8}$$

is an evolution triple, hence

$$L^2(0,T;H) \subset L^2(0,T;V^\star) \subset L^2(0,T;H^\star).$$

Consequently, since

$$u'' \in L^2(0,T;V^\star), \text{ we have } (u')' \in L^2(0,T;H^\star)$$

which implies by $u' \in L^2(0,T;H)$ and (16.8)

$$u' \in C([0,T];V^\star).$$

Since $u_1 \in H \subset V^\star$, the initial condition $u(0) = u_1$ makes sense.

*Proof of Theorem 16.1.* We apply Galerkin's method. Let $w_1, w_2, \dots$ be a linearly independent system in $V$ such that the linear combinations are dense in $V$. We want to find the $m$-th approximation of $u$ in the form

$$u_m(t) = \sum_{l=1}^{m} g_{lm}(t) w_l \text{ where } g_{lm} \in W^{2,2}(0,T) = H^2(0,T) \tag{16.9}$$

such that for all $j = 1, ..., m$

$$\langle u_m''(t), w_j\rangle + \langle (Qu_m)(t), w_j\rangle + \tag{16.10}$$

$$\langle \varphi(x)h'(u_m), w_j\rangle + \langle \psi(t,x;u_m)u_m'(t), w_j\rangle = \langle F(t), w_j\rangle,$$

$$u_m(0) = u_{m0}, \quad u_m'(0) = u_{m1} \tag{16.11}$$

where $u_{m0}, u_{m1}$ are linear combinations of $w_1, w_2, ...$ satisfying

$$(u_{m0}) \to u_0 \text{ in } V, \quad (u_{m1}) \to u_1 \text{ in } H. \tag{16.12}$$

By the existence theorem for a system of functional differential equations with Carathéodory conditions (see [32]) there exists a solution of (16.10), (16.11) in a neighborhood of 0. The maximal solution of (16.10), (16.11) is defined in $[0, T]$. Indeed, multiplying (16.10) by $g_{jm}'(t)$ and taking the sum with respect to $j$, we obtain

$$\langle u_m''(t), u_m'(t)\rangle + \langle (Qu_m)(t), u_m'(t)\rangle +$$

$$\langle \varphi(x)h'(u_m), u_m'(t)\rangle + \langle \psi(t,x;u_m)u_m'(t), u_m'(t)\rangle = \langle F(t), u_m'(t)\rangle.$$

Integrate the above equality over $[0, t]$, we find by (14.8), Remark 6.7 and Young's inequality

$$\frac{1}{2}\|u_m'(t)\|_H^2 + \frac{1}{2}\langle (Qu_m)(t), u_m(t)\rangle + \tag{16.13}$$

$$\int_\Omega \varphi(x)h(u_m(t))dx + \int_0^t \int_\Omega \psi(\tau,x;u_m)[u_m'(\tau)]^2 dx d\tau =$$

$$\int_0^t \langle F(\tau), u_m'(\tau)\rangle d\tau + \frac{1}{2}\|u_m'(0)\|_H^2 + \frac{1}{2}\langle Qu_m(0), u_m(0)\rangle +$$

$$\int_\Omega \varphi(x)h(u_m(0))dx \le \frac{1}{2}\int_0^T \|F(\tau)\|_H^2 d\tau + \frac{1}{2}\int_0^t \|u_m'(\tau)\|_H^2 d\tau + \text{const}$$

where the constant is not depending on $m$ and $t$, because of $(D2)$, (16.3), (16.4), (16.12)

$$|h(\eta)| \le \text{const}(1 + |\eta|^{\varrho+2})$$

and by Sobolev's imbedding theorem (see [1] and also Theorem 4.17), $L^{\varrho+2}(\Omega)$ is continuously imbedded into $W^{1,2}(\Omega)$ since

$$\varrho + 2 \le \frac{2}{n-2} + 2 = \frac{2n-2}{n-2} < \frac{2n}{n-2}.$$

By $(D2)$, (16.3), (16.4), (16.13) implies

$$\|u_m'(t)\|_H^2 \le c_4 \left[1 + \int_0^t \|u_m'(\tau)\|_H^2 d\tau\right] \tag{16.14}$$

with some constant $c_4$, not depending on $t$ and $m$ (but depending on $\|u_0\|_V$, $\|u_1\|_H$, $\|F\|_{L^2(Q_T)}$.) Thus by Gronwall's inequality

$$\|u_m'(t)\|_H^2 \leq \text{const }, t \in [0, T]. \tag{16.15}$$

The constant is not depending on $t$ and $m$ (but depending on $\|u_0\|_V$, $\|u_1\|_H$, $\|F\|_{L^2(Q_T)}$). Thus (16.1), (16.13) imply

$$\|u_m(t)\|_V^2 \leq \text{const }, t \in [0, T]. \tag{16.16}$$

By (16.15), (16.16) the maximal solution $u_m$ of (16.10), (16.11) is defined on $[0, T]$ and $(u_m)$ is bounded in $L^\infty(0, T; V)$, $(u_m')$ is bounded in $L^\infty(0, T; H)$.

Consequently, there are a subsequence of $(u_m)$, again denoted by $(u_m)$, and $u \in L^\infty(0, T; V)$ such that

$$(u_m) \to u \text{ weakly in } L^\infty(0, T; V), \tag{16.17}$$

$$(u_m') \to u' \text{ weakly in } L^\infty(0, T; H), \tag{16.18}$$

which means that for any fixed $g \in L^1(0, T; V^\star)$ and $g_1 \in L^1(0, T; H)$

$$\int_0^T \langle g(t), u_m(t)\rangle dt \to \int_0^T \langle g(t), u(t)\rangle dt,$$

$$\int_0^T (g_1(t), u_m'(t))dt \to \int_0^T (g_1(t), u'(t))dt,$$

because $u_m, u$ (and $u_m', u'$) are linear continuous functionals on $L^1(0, T; V^\star)$ (and $L^1(0, T; H)$, respectively).

Since the imbedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact (if $\Omega$ is bounded and its boundary is "sufficiently good", see Theorem 4.1), by Theorem 10.1, (16.17), (16.18), for a subsequence

$$(u_m) \to u \text{ in } L^2(0, T; H) = L^2(Q_T) \text{ and a.e. in } Q_T. \tag{16.19}$$

As $\tilde{Q} : V \to V^\star$ is a linear and continuous operator, by (16.17), for all $v \in V$

$$\langle Qu_m(t), v\rangle \to \langle Qu(t), v\rangle \text{ weakly in } L^\infty(0, T) \tag{16.20}$$

and by (16.18)

$$\langle u_m''(t), v\rangle = \frac{d}{dt}\langle u_m'(t), v\rangle \to \langle u''(t), v\rangle \tag{16.21}$$

with respect to the weak convergence of the space of distributions $\mathcal{D}'(0, T)$.

Further, by (16.19) and the continuity of $h'$

$$\varphi(x)h'(u_m(t)) \to \varphi(x)h'(u(t)) \text{ for a.e. } (t, x) \in Q_T. \tag{16.22}$$

By (16.3), (16.4)

$$\|\varphi(x)h'(u_m(t))\|_{L^2(\Omega)} \leq \text{const}\|h'(u_m(t))\|_{L^2(\Omega)} \leq \tag{16.23}$$

$$\text{const}\left[\int_\Omega |u_m(t)|^{2(\varrho+1)}dx\right]^{1/2} = \text{const}\left[\int_\Omega |u_m(t)|^{q_0}dx\right]^{1/2} \le \text{const}\|u_m(t)\|_V^{q_0/2}$$

because for $2(\varrho+1) = q_0 \le \frac{2n}{n-2}$ we have by Sobolev's imbedding theorem (see, e.g [1] and also Theorem 4.17)

$$L^{q_0}(\Omega) \subset V \text{ since } \frac{1}{q_0} \ge \frac{1}{2} - \frac{1}{n} = \frac{n-2}{2n}.$$

Thus by the Cauchy–Schwarz inequality the sequence of functions $\varphi(x)h'(u_m(t))v$ is equiintegrable in $\Omega$ for each fixed $v \in V$ and a.a. $t \in [0,T]$. So by Vitali's theorem for a.a. $t \in [0,T]$

$$\lim_{m\to\infty}\int_\Omega \varphi(x)h'(u_m(t))vdx = \int_\Omega \varphi(x)h'(u(t))vdx. \tag{16.24}$$

Further, by the assumption of our theorem, for a.e. $t \in [0,T]$, for a subsequence

$$\psi(t,x;u_m) \to \psi(t,x;u) \text{ in } L^2(\Omega), \tag{16.25}$$

hence for all fixed $v \in V \subset H$, a.a. $t \in [0,T]$

$$\int_\Omega \psi(t,x;u_m)u_m'(t)vdx = \int_\Omega [\psi(t,x;u_m) - \psi(t,x;u)]u_m'(t)vdx + \tag{16.26}$$

$$\int_\Omega \psi(t,x;u)u_m'(t)vdx \to \int_\Omega \psi(t,x;u)u'(t)vdx$$

because for a.a. $t \in [0,T]$, $u_m'(t)$ is bounded in $L^2(\Omega)$ and

$$u_m'(t) \to u'(t) \text{ weakly in } H.$$

Let $v \in V$ be an arbitrary element and $v_N = \sum_{j=1}^N \beta_j w_j$ a sequence, approximating $v$ with respect to the norm of $V$. By (16.10) we have

$$\langle u_m''(t), v_N\rangle + \langle Qu_m(t), v_N\rangle + \langle\varphi(x)h'(u_m(t)), v_N\rangle +$$

$$\langle\psi(t,x;u_m)u_m'(t), v_N\rangle = \langle F(t), v_N\rangle$$

which implies as $N \to \infty$

$$\langle u_m''(t), v\rangle + \langle Qu_m(t), v\rangle + \langle\varphi(x)h'(u_m)(t), v\rangle +$$

$$\langle\psi(t,x;u_m)u_m'(t), v\rangle = \langle F(t), v\rangle \text{ for a.a. } t \in [0,T].$$

By using (16.20), (16.21), (16.24), (16.26) we obtain from the above equality as $m \to \infty$

$$\lim_{m\to\infty}\langle u_m''(t), v\rangle + \langle Qu(t), v\rangle + \langle\varphi(x)h'(u)(t), v\rangle + \tag{16.27}$$

$$\langle\psi(t,x;u)u'(t), v\rangle = \langle F(t), v\rangle.$$

Equality (16.27) means that for a.a. $t \in [0,T]$, $u_m''(t)$ is weakly converging to an element of $V^\star$ and this limit as a function of $t$ belongs to $L^2(0,T;V^\star)$. Thus $u'' \in L^2(0,T;V^\star)$ and it is not difficult to show that

$$(u_m'') \to u'' \text{ weakly in } L^2(0,T;V^\star). \tag{16.28}$$

According to (16.17), (16.18) $u \in L^\infty(0,T;V)$, $u' \in L^\infty(0,T;H)$ thus Theorem 6.6 implies $u \in C(0,T;H)$ and for $\psi \in C^\infty[0,T]$ with the property $\psi(0) = 1$, $\psi(T) = 0$ we have for all $j$

$$\int_0^T \langle u'(t), w_j\rangle \psi(t)dt = -\langle u(0), w_j\rangle - \int_0^T \langle u(t), w_j\rangle \psi'(t)dt,$$

$$\int_0^T \langle u_m'(t), w_j\rangle \psi(t)dt = -\langle u_m(0), w_j\rangle - \int_0^T \langle u_m(t), w_j\rangle \psi'(t)dt.$$

Hence by (16.11), (16.12), (16.17), (16.18) we obtain as $m \to \infty$

$$\langle u_0, w_j\rangle = \lim_{m\to\infty} \langle u_{m0}, w_j\rangle = \lim_{m\to\infty} \langle u_m(0), w_j\rangle = \langle u(0), w_j\rangle$$

for all $j$ which implies $u(0) = u_0$.

Similarly, since $u' \in L^\infty(0,T;H)$, $u'' \in L^2((0,T;V^\star)$, by using Remark 16.2, we obtain $u'(0) = u_1$ and so by (16.27) Theorem 16.1 is proved. □

## Uniqueness and smoothness of solutions

Now we formulate and prove a theorem on the uniqueness and continuous dependence of the solution on $F$, $u_0$, $u_1$.

**Theorem 16.3.** *Assume that the conditions of Theorem 16.1 are fulfilled so that $\psi(t,x;u) = \tilde{\psi}(x)$ with the property*

$$0 \le \tilde{\psi}(x) \le \mathit{const}, \tag{16.29}$$

*$h''$ is continuous and satisfies*

$$|h''(\eta)| \le \mathit{const}|\eta|^\varrho. \tag{16.30}$$

*Then the solution of (16.6), (16.7) is unique. Further, if $u_j$ is a solution of (16.6), (16.7) with $F = F_j$, $u_0 = u_0^j$, $u_1 = u_1^j$ $(j = 1,2)$ then for*

$$w = u_1 - u_2 \text{ and } w_1(s) = \int_0^s [u_1(\tau) - u_2(\tau)]d\tau$$

*we have*

$$\|w(s)\|_H^2 + \|w_1(s)\|_V^2 \le \tag{16.31}$$
$$\chi_0(F_j, u_0^j, u_1^j)e^s \left[\|f_1 - f_2\|^2_{L^2(Q_s)} + \|u_0^1 - u_0^2\|_H^2 + \|u_1^1 - u_1^2\|_V^2\right]$$

*where* $\chi_0$ *is a function, the values of which are bounded if* $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $\|u_1^j\|_H$ *are bounded and*

$$f_j(t) = \int_0^t F_j(\tau)d\tau.$$

*Proof.* Assume that $u_j$ is a solution of (16.6), (16.7) with $F = F_j$, $u_0 = u_0^j$, $u_1 = u_1^j$ $(j = 1, 2)$. Let $s \in [0, T]$ be an arbitrary fixed number and apply (16.6) (with $u_j$) to $v$, defined by

$$v(t) = \int_t^s [u_1(\tau) - u_2(\tau)]d\tau \text{ if } 0 \le t \le s \text{ and}$$

$$v(t) = 0 \text{ if } s < t \le T.$$

It is not difficult to show that

$$v \in C(0, T; V), \quad v' \in L^\infty(0, T; V), \tag{16.32}$$

$$v'(t) = -w(t) = u_2(t) - u_1(t) \text{ if } t < s \text{ and } v'(t) = 0 \text{ if } t > s$$

and thus

$$\langle w''(t), v(t)\rangle + \langle Qw(t), v(t)\rangle + \langle \varphi(x)[h'(u_1(t)) - h'(u_2(t))], v(t)\rangle +$$
$$\langle \tilde{\psi}(x)w'(t), v(t)\rangle = \langle F_1(t) - F_2(t), v(t)\rangle.$$

Integrating over $(0, s)$, by (16.32) we obtain

$$\int_0^s \langle w''(t), v(t)\rangle dt - \int_0^s \langle Qv'(t), v(t)\rangle dt + \int_0^s \langle \tilde{\psi}(x)w'(t), v(t)\rangle dt = \tag{16.33}$$
$$\int_0^s \langle F_1(t) - F_2(t), v(t)\rangle dt - \int_0^s \langle \varphi(x)[h'(u_1(t)) - h'(u_2(t))], v(t)\rangle dt.$$

By Remarks 6.7, 16.2 and (16.32)

$$\int_0^s \langle w''(t), v(t)\rangle dt = \int_0^s \langle w'(t), w(t)\rangle dt - \langle w'(0), v(0)\rangle =$$
$$\frac{1}{2}\|w(s)\|_H^2 - \frac{1}{2}\|w(0)\|_H^2 - \langle w'(0), v(0)\rangle.$$

Since $v(s) = 0$, integrating by parts, by (14.8) we get from (16.33)

$$\frac{1}{2}\|w(s)\|_H^2 + \frac{1}{2}\langle Qv(0), v(0)\rangle + \int_0^s \left[\int_\Omega \tilde{\psi}(x)w^2(t)dx\right] dt = \tag{16.34}$$
$$\int_0^s \langle F_1(t) - F_2(t), v(t)\rangle + \int_\Omega w'(0)v(0)dx + \int_\Omega \tilde{\psi}(x)w(0)v(0)dx -$$
$$\int_0^s \langle \varphi(x)[h'(u_1(t)) - h'(u_2(t))], v(t)\rangle dt + \frac{1}{2}\|w(0)\|_H^2.$$

By using the definition of $w$ and the notation $w_1(s) = \int_0^s w(\tau)d\tau$ we have

$$v(0) = \int_0^s w(\tau)d\tau = w_1(s) \tag{16.35}$$

and so by (16.1)

$$\langle Qv(0), v(0)\rangle \geq c_0 \|v(0)\|_V^2 = c_0 \|w_1(s)\|_V^2. \tag{16.36}$$

By using the notation $f_j(t) = \int_0^t F_j(\tau)d\tau$, we obtain by integration by parts and Young's inequality

$$\Big|\int_0^s \langle F_1(t) - F_2(t), v(t)\rangle dt\Big| = \Big|\int_\Omega \left\{\int_0^s [f_1'(t) - f_2'(t)]v(t)dt\right\} dx\Big| = \tag{16.37}$$

$$\Big|\int_\Omega \left\{\int_0^s [f_1(t) - f_2(t)]w(t)dt\right\} dx\Big| \leq \frac{1}{2}\int_0^s \|w(t)\|_H^2 dt + \frac{1}{2}\|f_1 - f_2\|^2_{L^2(Q_s)}.$$

Similarly, by (16.35)

$$\Big|\int_\Omega w'(0)v(0)dx\Big| \leq \varepsilon \|w_1(s)\|_V^2 dt + C_2(\varepsilon)\|w'(0)\|_H^2 \tag{16.38}$$

and by (16.3)

$$\Big|\int_\Omega \tilde{\psi}(x)w(0)v(0)dx\Big| \leq \varepsilon \|w_1(s)\|_V^2 dt + C_3(\varepsilon)\|w(0)\|_H^2. \tag{16.39}$$

($C_j(\varepsilon)$ denote constants, depending on $\varepsilon$.)

Finally, the last term on the right hand side of (16.34) can be estimated as follows: by (16.3), (16.30) and Lagrange's mean value theorem

$$\Big|\int_0^s \langle \varphi(x)[h'(u_1(t)) - h'(u_2(t))], v(t)\rangle dt\Big| \leq \tag{16.40}$$

$$\text{const}\int_0^s \left\{\int_\Omega |h'(u_1(t)) - h'(u_2(t))||v(t)|dx\right\} dt =$$

$$\text{const}\int_0^s \left\{\int_\Omega \sup\{|h''(\eta)| : \eta \in (a,b)\}|u_1(t) - u_2(t)||v(t)|dx\right\} dt \leq$$

$$\text{const}\int_0^s \left\{\int_\Omega [|u_1(t)|^\varrho + |u_2(t)|^\varrho]|u_1(t) - u_2(t)||v(t)|dx\right\} dt$$

where

$$a = \min\{u_1(t), u_2(t)\}, \quad b = \max\{u_1(t), u_2(t)\}.$$

Since

$$\varrho n \leq \frac{2n}{n-2} = q,$$

$V$ is continuously imbedded into $L^{\rho n}(\Omega)$ and $L^q(\Omega)$, and so we may apply Hölder's inequality by $\frac{1}{n} + \frac{1}{2} + \frac{1}{q} = 1$:

$$\int_0^s \left\{ \int_\Omega [|u_1(t)|^\varrho + |u_2(t)|^\varrho]|w(t)||v(t)|dx \right\} dt \leq \tag{16.41}$$

$$\text{const} \int_0^s \left[ \||u_1(t)|^\varrho\|_{L^n(\Omega)} + \||u_2(t)|^\varrho\|_{L^n(\Omega)} \right] \|w(t)\|_H \|v(t)\|_{L^q(\Omega)} dt.$$

$$\text{const} \int_0^s [\|u_1(t)\|_V^\varrho + \|u_2(t)\|_V^\varrho] \|w(t)\|_H \|v(t)\|_{L^q(\Omega)} dt$$

Since $u_1, u_2 \in L^\infty(0,T;V)$ and according to the proof of Theorem 16.1, their $L^\infty(0,T;V)$ norm can be estimated by a function of $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $\|u_1^j\|_H$, the values of which are bounded if $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $\|u_1^j\|_H$ are bounded (see (16.14) – (16.16)), we obtain from (16.40), (16.41) and $v(t) = w_1(s) - w_1(t)$ (for $t \leq s$) that

$$\left| \int_0^s \langle \varphi(x)[h'(u_1(t)) - h'(u_2(t))], v(t) \rangle dt \right| \leq \tag{16.42}$$

$$\chi(F_j, u_0^j, u_1^j) \int_0^s \|w(t)\|_H \|v(t)\|_{L^q(\Omega)} dt \leq$$

$$\chi(F_j, u_0^j, u_1^j) \int_0^s \|w(t)\|_H \left[ \|w_1(t)\|_{L^q(\Omega)} + \|w_1(s)\|_{L^q(\Omega)} \right] dt \leq$$

$$\chi(F_j, u_0^j, u_1^j) \left[ \varepsilon \|w_1(s)\|_V^2 + C(\varepsilon) \int_0^s \left( \|w(t)\|_H^2 + \|w_1(t)\|_V^2 \right) dt \right]$$

where $\chi(F_j, u_0^j, u_1^j)$ is bounded if $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $\|u_1^j\|_H$ are bounded.

Choosing sufficiently small $\varepsilon > 0$, we obtain from (16.34), (16.36) – (16.39), (16.42) with some $\tilde{\chi}(F_j, u_0^j, u_1^j)$

$$\|w(s)\|_H^2 + \|w_1(s)\|_V^2 \leq \tilde{\chi}(F_j, u_0^j, u_1^j) \int_0^s [\|w(t)_H^2 + \|w_1(t)\|_V^2] dt +$$

$$c_6 \left[ \|f_1 - f_2\|_{L^2(Q_s)}^2 + \|w(0)\|_H^2 + \|w'(0)\|_V^2 \right].$$

Hence by Gronwall's lemma

$$\|w(s)\|_H^2 + \|w_1(s)\|_V^2 \leq$$

$$\chi_0(F_j, u_0^j, u_1^j) \mathrm{e}^s \left[ \|f_1 - f_2\|_{L^2(Q_s)}^2 + \|w(0)\|_H^2 + \|w'(0)\|_V^2 \right].$$

Thus we have (16.31) and, consequently, the uniqueness of the solution of (16.6), (16.7). $\square$

If $F$, $u_0$, $u_1$ satisfy certain smoothness conditions then we have smoother solutions of (16.6), (16.7).

**Theorem 16.4.** *Assume that the conditions of Theorem 16.3 are fulfilled so that the restriction of (the linear and continuous operator) $\tilde{Q} : V \to V^\star$ to $V \cap H^2(\Omega)$ is continuous operator from $H^2(\Omega)$ into $H = L^2(\Omega)$;*

$$F' \in L^2(Q_T), \quad u_0 \in V \cap H^2(\Omega), \quad u_1 \in V. \tag{16.43}$$

*Then there exists a (unique) solution*

$$u \in L^\infty(0, T; V) \tag{16.44}$$

*of (16.6), (16.7) satisfying*

$$u' \in L^\infty(0, T; V), \quad u'' \in L^\infty(0, T; H). \tag{16.45}$$

*If $\tilde{Q} : V \to V^\star$ is such that for any*

$$f \in L^2(\Omega), \quad \tilde{Q}\tilde{u} = f \text{ imply } \tilde{u} \in H^2(\Omega) \text{ and} \tag{16.46}$$

$$\|\tilde{u}\|_{H^2(\Omega)} \le const\|f\|_{L^2(\Omega)}$$

*then for the solution $u$ of (16.6), (16.7) we have*

$$u \in L^\infty(0, T; V \cap H^2(\Omega)). \tag{16.47}$$

*Proof.* We apply Galerkin's method and, similarly to the proof of Theorem 16.1, we want to find the solution $u$ of (16.6), (16.7) as the limit of functions $u_m$ given by (16.9) with $g_{lm} \in H^3(0, T)$, satisfying (16.10), (16.11) and instead of (16.12) we have

$$(u_{m0}) \to u_0 \text{ in } V \cap H^2(\Omega), \quad (u_{m1}) \to u_1 \text{ in } V. \tag{16.48}$$

Since $h''$ is continuous, we may differentiate (16.10) with respect to $t$, so we obtain

$$\langle u_m^{(3)}(t), w_j\rangle + \langle Qu_m'(t), w_j\rangle + \langle \varphi(x)h''(u_m)u_m'(t), w_j\rangle + \tag{16.49}$$
$$\langle \tilde{\psi}(x)u_m''(t), w_j\rangle = \langle F'(t), w_j\rangle.$$

Multiplying (16.49) with $g_{jm}''(t)$ and taking the sum with respect to $j$, we find

$$\langle u_m^{(3)}(t), u_m''(t)\rangle + \langle Qu_m'(t), u_m''(t)\rangle + \langle \varphi(x)h''(u_m)u_m'(t), u_m''(t)\rangle + \tag{16.50}$$
$$\langle \tilde{\psi}(x)u_m''(t), u_m''(t)\rangle = \langle F'(t), u_m''(t)\rangle.$$

Integrating both sides of (16.50) over $(0, t)$, we obtain (similarly to (16.13))

$$\frac{1}{2}\|u_m''(t)\|_H^2 + \frac{1}{2}\langle Qu_m'(t), u_m'(t)\rangle + \tag{16.51}$$

$$\int_0^t \int_\Omega \varphi(x)h''(u_m)u_m'(\tau)u_m''(\tau)dxd\tau + \int_0^t \int_\Omega \tilde{\psi}(x)[u_m''(\tau)]^2 dxd\tau x =$$

$$\int_0^t \langle F'(\tau), u_m''(\tau)\rangle d\tau + \frac{1}{2}\|u_m''(0)\|_H^2 + \frac{1}{2}\langle Qu_m'(0), u_m'(0)\rangle.$$

Further, multiplying (16.10) by $g_{jm}''(t)$ and summing with respect to $j$, we obtain

$$\|u_m''(t)\|_H^2 + \langle Qu_m(t), u_m''(t)\rangle + \langle \varphi(x)h'(u_m), u_m''(t)\rangle +$$
$$\langle \tilde{\psi}(x)u_m'(t), u_m''(t)\rangle = \langle F(t), u_m''(t)\rangle,$$

thus

$$\|u_m''(0)\|_H^2 \le$$
$$\left[\|F(0)\|_H + \|\tilde{Q}u_m(0)\|_H + c_2\|h'(u_m(0))\|_H + c_3\|u_m'(0)\|_H\right] \|u_m''(0)\|_H.$$

So by (16.48) and Sobolev's imbedding theorem (see (16.23))

$$\|u_m''(0)\|_H \le \text{const for all } m \tag{16.52}$$

since by the assumption of our theorem

$$\|\tilde{Q}u_m(0)\|_H \le \text{const}\|u_m(0)\|_{H^2(\Omega)}.$$

Finally, the third term on the left hand side of (16.51) can be estimated as follows: (similarly to (16.41), (16.42)) by Hölder's inequality with $\frac{1}{n}+\frac{1}{2}+\frac{1}{q}=1$

$$\left|\int_0^t \left[\int_\Omega \varphi(x)h''(u_m(\tau))u_m'(\tau)u_m''(\tau)dx\right] d\tau\right| \le \tag{16.53}$$
$$\text{const}\int_0^t \left[\int_\Omega |u_m(\tau)|^\varrho |u_m'(\tau)||u_m''(\tau)|dx\right] d\tau \le$$
$$\text{const}\int_0^t \left[\|u_m(\tau)^\varrho\|_{L^n(\Omega)}\|u_m'(\tau)\|_{L^q(\Omega)}\|u_m''(\tau)\|_H\right] d\tau \le$$
$$\text{const}\int_0^t \left[\|u_m(\tau)\|_V^\varrho\|u_m'(\tau)\|_V\|u_m''(\tau)\|_H\right] d\tau \le$$
$$\text{const}\int_0^t \|u_m'(\tau)\|_V\|u_m''(\tau)\|_H d\tau \le \text{const}\int_0^t \left[\|u_m'(\tau)\|_V^2 + \|u_m''(\tau)\|_H^2\right] d\tau$$

since $(u_m)$ is bounded in $L^\infty(0,T;V)$.

Thus, (16.51) - (16.53) (16.1), (ii) and Young's inequality imply

$$\|u_m''(t)\|_H^2 + \|u_m'(t)\|_V^2 \le \text{const}\left\{1 + \int_0^t \left[\|u_m''(\tau)\|_H^2 + \|u_m'(\tau)\|_V^2\right] d\tau\right\}$$

and so by Gronwall's lemma for all $m$, $t \in [0,T]$

$$\|u_m''(t)\|_H^2 + \|u_m'(t)\|_V^2 \le \text{const}. \tag{16.54}$$

Hence, similarly to the proof of Theorem 16.1 we obtain

$$(u_m') \to u' \text{ weakly in } L^\infty(0,T;V),$$

$$(u''_m) \to u'' \text{ weakly in } L^\infty(0,T;H),$$

we have (16.17), too, for the (unique) solution of (16.6), (16.7).

If (16.46) holds then from the equation (16.6) and (16.44), (16.45) we obtain directly (16.47). □

**Remark 16.5.** According to [51] the operator $\tilde{Q}$ given in (14.29) satisfies (16.46).

## Solutions in $(0.\infty)$

Similarly to the previous existence theorems, one can prove existence of solutions to (16.6), (16.7) for $t \in (0,\infty)$.

**Theorem 16.6.** *Assume that the conditions of Theorem 16.1 are fulfilled for all $T > 0$ with*

$$\psi : Q_\infty \times L^2(Q_\infty) \to \mathbb{R},$$

*satisfying (16.3) for all $t \in (0,\infty)$. Then for any $F \in L^2_{loc}(0,\infty;H)$, $u_0 \in V$, $u_1 \in H$ there exists $u \in L^\infty_{loc}(0,\infty;V)$ such that $u' \in L^\infty_{loc}(0,\infty;H)$, $u'' \in L^2_{loc}(0,\infty;V^\star)$ and for a.a. $t \in (0,\infty)$, (16.6) and (16.7) hold.*

**Theorem 16.7.** *Assume that the conditions of Theorem 16.4 are fulfilled for all finite $T > 0$ and the conditions of Theorem 16.6 are satisfied, too. If*

$$F' \in L^2_{loc}(0,\infty;H), \quad u_0 \in V \cap H^2(\Omega), \quad u_1 \in V$$

*then there exists a (unique) solution of (16.6), (16.7)*

$$u \in L^\infty_{loc}(0,\infty;V) \text{ satisfying } u' \in L^\infty_{loc}(0,\infty;V), \quad u'' \in L^\infty_{loc}(0,\infty;H).$$

*Further, (16.46) implies $u \in L^\infty_{loc}(0,\infty;V \cap H^2(\Omega))$.*

Now we formulate and prove a theorem on the "boundedness" of the solution of (16.6), (16.7) for $t \in (0,\infty)$.

**Theorem 16.8.** *Assume that the conditions of Theorem 16.7 are fulfilled such that $\tilde{\psi}(x) \geq \tilde{c} \geq 0$, on $F$ assuming only $F \in L^2_{loc}(0,\infty;H)$ and $u$ is a solution of (16.6), (16.7) for $t \in (0,\infty)$.*

*If with some $T_0 > 0$, $F(t) = 0$ for a.a. $t > T_0$ then*

$$\|u'(t)\|_H^2 + c_0\|u(t)\|_V^2 + 2c_1 \int_\Omega h(u(t))dx + \tag{16.55}$$

$$\tilde{c}\int_0^t \left[\int_\Omega |u'(\tau)|^2 dx\right] d\tau \leq const, \quad t \in (0,\infty).$$

*Consequently,*

$$u \in L^\infty(0,\infty;V), \quad u' \in L^\infty(0,\infty;H) \text{ and } \tilde{\psi}^{1/2}u' \in L^2(0,\infty;H).$$

*Further,*

$$F \in L^2(0, \infty; H) \text{ and } \tilde{c} > 0 \tag{16.56}$$

*also imply (16.55). Consequently,*

$$\|u'(t)\|_H \leq const\ e^{-\tilde{c}t}, \quad t \in (0, \infty). \tag{16.57}$$

*Finally, if* $F \in L^\infty(0, \infty; H)$ *and* $\tilde{c} > 0$ *then*

$$\frac{1}{t} \int_0^t \|u'(\tau)\|_H^2 d\tau \leq const$$

*and thus*

$$\|u'(t)\|_H^2 + c_0 \|u(t)\|_V^2 + 2c_1 \int_\Omega h(u(t))dx + 2\tilde{c} \int_0^t \left[ \int_\Omega |u'(\tau)|^2 dx \right] d\tau \leq \hat{c}t \tag{16.58}$$

*with some constant* $\hat{c}$.

*Proof.* Let $u$ be a solution of (16.6), (16.7) for $t \in (0, \infty)$. By (16.1), (16.13) we obtain for a.a. $t \in (0, \infty)$

$$\frac{1}{2}\|u_m'(t)\|_H^2 + \frac{1}{2}c_0\|u_m(t)\|_V^2 + \int_\Omega \varphi(x)h(u_m(t))dx + \tag{16.59}$$

$$\tilde{c} \int_0^t \left[ \int_\Omega |u_m'(\tau)|^2 dx \right] d\tau \leq$$

$$\frac{1}{2} \int_0^{T_0} \|F(\tau)\|_H^2 d\tau + \frac{1}{2} \int_0^{T_0} \|u_m'(\tau)\|_H^2 d\tau + \hat{c}$$

where the constant $\hat{c}$ is independent of $t$. Thus

$$\|u_m'(t)\|_H^2 \leq c_4 + \int_0^t b(\tau)\|u_m'(\tau)\|_H^2 d\tau \tag{16.60}$$

with some constant $c_4$ (independent of $t$) and

$$b(\tau) = 1 \text{ for } 0 \leq \tau \leq T_0, \quad b(\tau) = 0 \text{ for } \tau > T_0.$$

From (16.60) by Gronwall's lemma we find

$$\|u_m'(t)\|_H^2 \leq c_4 + c_4 \int_0^{T_0} \mathrm{e}^{T_0 - s} ds \leq \text{const}, \quad t \in (0, T_0). \tag{16.61}$$

Since by (16.17), (16.18)

$$\|u(t)\|_V^2 \leq \liminf \|u_m(t)\|_V^2, \quad \|u'(t)\|_H^2 \leq \liminf \|u_m'(t)\|_H^2,$$

by (16.19), Vitali's theorem, from (16.59) we obtain as $m \to \infty$ (16.55).

If $F \in L^2(0,\infty;H)$ and $\tilde{c} > 0$, we obtain similarly to (16.59)

$$\frac{1}{2}\|u_m'(t)\|_H^2 + \frac{1}{2}c_0\|u_m(t)\|_V^2 + \int_\Omega \varphi(x)h(u_m(t))dx + \tag{16.62}$$

$$\tilde{c}\int_0^t \left[\int_\Omega |u_m'(\tau)|^2 dx\right] d\tau \le \varepsilon \int_0^t \|u_m'(\tau)\|_H^2 d\tau + C(\varepsilon)\int_0^t \|F(\tau)\|_H^2 d\tau + \hat{c}.$$

By (16.56) and (16.62) with sufficiently small $\varepsilon > 0$, we obtain

$$\int_0^t \|u_m'(\tau)\|_H^2 d\tau \le \text{const}, \quad t \in (0,\infty)$$

and so we obtain from (16.62) (16.55). Further, by (16.55)

$$\|u'(t)\|_H^2 + 2\tilde{c}\int_0^t \|u'(\tau)\|_H^2 d\tau \le c^\star$$

with some positive constant $c^\star$. Thus by using Gronwall's lemma we obtain

$$\|u'(t)\|_H^2 \le c^\star \mathrm{e}^{-2\tilde{c}t}$$

which implies (16.57).

Finally, if $F \in L^\infty(0,\infty;H)$ and $\tilde{c} > 0$, we have similarly to (16.62),

$$\frac{1}{2}\|u_m'(t)\|_H^2 + \frac{1}{2}c_0\|u_m(t)\|_V^2 + \int_\Omega \varphi(x)h(u_m(t))dx + \tag{16.63}$$

$$\tilde{c}\int_0^t \left[\int_\Omega |u_m'(\tau)|^2 dx\right] d\tau \le \|F\|_{L^\infty(0,\infty;H)} \int_0^t \|u_m'(\tau)\|_H d\tau \le$$

$$\text{const} \cdot t^{1/2} \left[\int_0^t \|u_m'(\tau)\|_H^2 d\tau\right]^{1/2}.$$

By using the notation

$$Y(t) = \int_0^t \|u_m'(\tau)\|_H^2 d\tau, \text{ we have } Y'(t) = \|u_m'(t)\|_H^2$$

and thus

$$Y'(t) + c_4 Y(t) \le c_5 t^{1/2}[Y(t)]^{1/2} + c_6 \tag{16.64}$$

with constants $c_4, c_5, c_6 > 0$. Set $z(t) = Y(t)/t$, then from (16.64) we obtain

$$z'(t) + \left(c_4 + \frac{1}{t}\right) z(t) \le c_5[z(t)]^{1/2} + \frac{c_6}{t},$$

whence

$$z'(t) + c_4 z(t) \le c_5[z(t)]^{1/2} + \frac{c_6}{t} \le 2c_5[z(t)]^{1/2}$$

if $t \geq c_5 c_6 = t_0$ and $z(t) \geq 1$. Thus, assuming $z(t) \geq 1$, we obtain for $t > t_0$

$$z'(t) \leq d_1[z(t)]^{1/2} - d_2 z(t) \text{ where } d_2 > 0. \tag{16.65}$$

Inequality (16.65) implies that $z(t)$ is bounded for $t > t_0$ because if $z(t) > (d_1/d_2)^2$ then the right hand side of (16.65) is negative, thus the nonnegative function $z$ is decreasing.

Consequently, there is a constant $c^\star$ such that

$$0 \leq z(t) \leq c^\star, \quad t \in (0, \infty), \text{ i.e.}$$

$$\frac{1}{t}\int_0^t \|u'(\tau)\|_H^2 d\tau \leq c^\star$$

and by (16.63) we have (16.58). □

**Remark 16.9.** Assume that the conditions of Theorem 16.8 are fulfilled in the following form: there exist $F_\infty \in H$ and $u_\infty \in V$ such that

$$F - F_\infty \in L^2(0, \infty; H) \text{ and}$$

$$u_\infty \in V \text{ is a solution of } \tilde{Q}u_\infty = F_\infty. \tag{16.66}$$

(Such $u_\infty \in V$ exists if $\tilde{Q}$ is an elliptic operator with $K = 0$, considered in Theorem 14.5) Then (16.55) holds. Indeed, taking the difference of (**??**) and (16.66), we obtain (16.62) with $w = u - u_\infty$, instead of $u$ (in the third term on the left hand side with $u$) and with $F - F_\infty$, instead of $F$.

**Theorem 16.10.** *Assume that the conditions of Theorem 16.8 are satisfied in the more general form, formulated in Remark 16.9, i.e. there exists $F_\infty \in H$ such that $F - F_\infty \in L^2(0, \infty; H)$ and $u_\infty$ is a solution of (16.66), i.e. $\tilde{Q}u_\infty = F_\infty$. Further,*

$$\tilde{\psi}(x) \geq \tilde{c} \text{ with a constant } \tilde{c} > 0. \tag{16.67}$$

*Then for the solution $u$ of (16.6), (16.7)*

$$\int_0^\infty \|u'(\tau)\|_H d\tau < \infty, \text{ i.e. } u' \in L^1(0, \infty; H)$$

*and there exists $w_0 \in H$ such that*

$$u(T) \to w_0 \text{ in } H \text{ as } T \to \infty, \quad \|u(T) - w_0\|_H \leq const\ e^{-\tilde{c}T}.$$

*Proof.* According to Theorem 16.8

$$\|u'(t)\|_H \leq \text{const e}^{-\tilde{c}t}, \tag{16.68}$$

thus

$$\int_0^\infty \|u'(t)\|_H d\tau < \infty. \tag{16.69}$$

Further, applying Theorem 6.6 to $v = u(T_2) - u(T_1)$ (which is constant in $t$), we obtain by using $u'(t) \in H$

$$\|u(T_2) - u(T_1)\|_H^2 = (u(T_2), u(T_2) - u(T_1)) - (u(T_1), u(T_2) - u(T_1)) =$$
$$\int_{T_1}^{T_2} \langle u'(t), u(T_2) - u(T_1)\rangle dt = \int_{T_1}^{T_2} (u'(t), u(T_2) - u(T_1))dt \le$$
$$\|u(T_2) - u(T_1)\|_H \int_{T_1}^{T_2} \|u'(t)\|_H dt$$

which implies

$$\|u(T_2) - u(T_1)\|_H \le \int_{T_1}^{T_2} \|u'(t)\|_H dt.$$

Hence by (16.69)

$$\|u(T_2) - u(T_1)\|_H \to 0$$

as $T_1, T_2 \to \infty$, i.e. there is some $w_0 \in H$ such that

$$u(T) \to w_0 \text{ in } H \text{ as } T \to \infty$$

and by (16.68)

$$\|u(T) - w\|_H \le \int_T^\infty \|u'(t)\|_H dt \le \text{const} \int_T^\infty \mathrm{e}^{-\tilde{c}t} dt = \text{const } \mathrm{e}^{-\tilde{c}T}.$$

□

**Theorem 16.11.** *Assume that the conditions of Theorem 16.10 are satisfied, further,* $F' \in L^2(0, \infty; H)$, $h \in C^2$ *and* $h''$ *is bounded. Then*

$$\|u''(t)\|_H \le \textit{const } e^{-\tilde{c}t}, \quad t \in (0, \infty) \textit{ and} \tag{16.70}$$
$$u' \in L^\infty(0, \infty; V).$$

*Further, if* $\lim_{t\to\infty} \|F(t) - F_\infty\|_H = 0$ *then for the function* $w_0$ *satisfying* $\lim_{t\to\infty} \|u(t) - w_0\|_H = 0$ *(see Theorem 16.10) we have with arbitrary* $\chi \in V \cap H^2(\Omega)$

$$\int_\Omega w_0 \tilde{Q}\chi dx + \int_\Omega \varphi(x) h(w_0(x))\chi dx = \int_\Omega F_\infty \chi dx. \tag{16.71}$$

*If* $\tilde{Q}$ *is defined by*

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \int_\Omega \left[\sum_{j,l=1}^n a_{jl}(x)(D_l\tilde{u})(D_j\tilde{v}) + d(x)\tilde{u}\tilde{v}\right] dx$$

*(see Theorem 14.5) then equation (16.71) means that* $w_0$ *is a weak (distributional) solution of*

$$-\sum_{j,l=1}^n D_j[a_{jl}(x)D_l w_0] + \varphi(x)h(w_0(x)) = F_\infty$$

*(with some homogeneous boundary conditions).*

*Sketch of the proof.* One applies the arguments in the proof of Theorem 16.4. Since $h''$ is bounded, the third term on the left hand side of (16.51) can be estimated as follows

$$\left| \int_0^t \left[ \int_\Omega \varphi(x) h''(u_m(\tau)) u_m'(\tau) u_m''(\tau) dx \right] d\tau \right| \leq$$

$$\text{const} \int_0^t \|u_m'(\tau)\|_H \|u_m''(\tau)\|_H d\tau \leq \varepsilon \int_0^t \|u_m''(\tau)\|_H^2 + C(\varepsilon) \int_0^t \|u_m'(\tau)\|_H^2 d\tau.$$

Choosing sufficiently small $\varepsilon > 0$, we obtain from (16.51)

$$\|u_m''(t)\|_H^2 + \tilde{c} \int_0^t \|u_m''(\tau)\|_H^2 d\tau \leq \text{const}, \quad t \in (0, \infty).$$

Thus Gronwall's lemma implies (16.70). Applying (16.6) to $\chi \in V \cap H^2(\Omega)$, we obtain (16.71) as $t \to \infty$.

$\square$

## Problems

1. Prove Theorem 16.6.

2. Prove Theorem 16.7.

3. Prove Theorem 16.11

# Bibliography


[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York - San Francisco - London, 1975.

[2] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I.*, Comm. Pure Appl. Math. **12** (1959), 623-727.

[3] H. Amann, *Semigroups and nonlinear evolution equations*, Linear Algebra and its Appl. **84** (1986), 3-32.

[4] H. Amann, *Parabolic evolution equations and nonlinear boundary conditions*, J. Diff. Eqns. **72** (1988), 201-269.

[5] D. Arcoya, J. I. Díaz and L. Tello, *S-sharped bifurcation branch in a quasilinear multivalued model arising in climatology*, J. Diff. Eqns. **150** (1998), 215-225.

[6] M. Badii, J. I. Díaz and A. Tesei, *Existence and attractivity results for a class of degenerate functional parabolic problems*, Rend. Sem. Mat. Univ. Padova **78** (1987), 110-124.

[7] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, 1976.

[8] J. Berkovits and V. Mustonen, *Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems*, Rend. Mat. Ser. VII, **12**, Roma (1992), 597-621.

[9] Á. Besenyei, *On systems of parabolic functional differential equations*, Annales Univ. Sci. Budapest. Eötvös, Sect. Math., **47** (2004), 143-160.

[10] Á. Besenyei, *On a nonlinear system consisting of three different types of differential equations*, Acta Math. Hungar. **127** (2010), 178-194.

[11] H. Brézis, *Operateurs Maximaux Monotones et Semi-groups de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[12] H. Brézis and F. E. Browder, *Strongly nonlinear parabolic initial-boundary value problems*, Proc. Natl. Acad. Sci. USA **76** (1979), 38-40.

[13] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer, 1996.

[14] F. E. Browder, *Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains*, Proc. Natl. Acad. Sci. USA **74** (1977), 2659-2661.

[15] F. E. Browder, *Strongly nonlinear parabolic equations of higher order*, Atti Acc. Lincei **77** (1986), 159-172.

[16] M. Chipot and B. Lovat, *Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems, advances in quenching*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **8** (2001), 35-51.

[17] M. Chipot and L. Molinet, *Asymptotic behavior of some nonlocal diffusion problems*, Applicable Analysis **80** (2001), 279-315.

[18] S. Cinca, *Diffusion und Transport in porösen Medien bei veränderlichen Porosität*, Diplomawork, Univ. Heidelberg, 2000.

[19] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

[20] K. Deimling, *Nonlinear Functional Analysis*, Springer, 1985.

[21] J.I. Díaz, J. Hernández and L. Tello, *On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in climatology*, J. Math. Anal. Appl. **216** (1997), 593-613.

[22] J. I. Díaz and G. Hetzer, *A quasilinear functional reaction-diffusion equation arising in climatology*, in: Équations aux Dérivées Partielles et Applications, Gauthier-Villars, Paris (1998), 461-480.

[23] J. I. Díaz and F. de Thelin, *On a nonlinear parabolic problem arising in some models related to turbulent flows*, SIAM J. Math. Anal. **25** (1994), 1085-1111.

[24] O. Diekmann, S. A. van Gils, S. M. V. Lunel and H-O. Walther, *Delay Equations*, Springer, 1995.

[25] Yu. A. Dubinskiy, *Nonlinear elliptic and parabolic equations (in Russian)*, in: Modern Problems in Mathematics **9**, Moscow, 1976.

[26] D. E. Edmunds and J. R. L. Webb, *Quasilinear elliptic problems in unbounded domains*, Proc. Roy. Soc. London, Ser. A **337** (1973), 397-410.

[27] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, 1998.

[28] A. Friedman, *Partial Differential Equations of Parabolic Type*, Krieger, Malabar, FL, 1983.

[29] S. Fučik and A. Kufner, *Nonlinear Differential Equations*, Elsevier, Amsterdam, 1980.

[30] H. Gajewski, K. Gröger and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferenzialgleichungen*, Akademie-Verlag, Berlin, 1974.

[31] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1983.

[32] J. K. Hale, *Theory of Functional Differential Equations*, Springer, 1977.

[33] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Math. Soc., Providence, R.I., 1988.

[34] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer, 1993.

[35] P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, 1964.

[36] U. Hornung and W. Jäger, *Diffusion, absorption and reaction of chemicals in porous media*, J. Diff. Eqns. **92** (1991), 199-225.

[37] L. Hörmander, *Linear Partial Differential Operators*, Springer, 1963.

[38] U. Hornung, W. Jäger and A. Mikelič, *Reactive transport through an array of cells with semi-permeable membranes*, Math. Modelling Num. Anal. **28** (1994), 59-94.

[39] W. Jäger and N. Kutev, *Discontinuous solutions of the nonlinear transmission problem for quasilinear elliptic equations*, Preprint IWR der Univ. Heidelberg **98-22** (1998), 1-37.

[40] W. Jäger and L. Simon, *On transmission problems for nonlinear parabolic differential equations*, Annales Univ. Sci. Budapest **45** (2002), 143-158.

[41] W. Jäger and L. Simon, *On a system of quasilinear parabolic functional differential equations*, Acta Math. Hung. **112** (2006), 39-55.

[42] V. Komornik, *Lectures on Real Analysis I-II* (in French and Hungarian), Ellipses, 2001 and Typotex, 2003.

[43] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, American Math. Soc., Providence, R.I., 1968.

[44] O. A. Ladyzhenskaya, N. N. Uraltseva, *Linear ans Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[45] R. Landes, *Quasilinear elliptic operators and weak solutions of the Euler equation*, Manuscripta Math. **27** (1979), 47-72.

[46] R. Landes, *On Galerkin's method in the existence theory of quasilinear elliptic equations*, J. Functional Analysis **39** (1980), 123-148.

[47] R. Landes and V. Mustonen, *On pseudomonotone operators and nonlinear noncoercive variational problems on unbounded domains*, Math. Ann. **248** (1980), 241-246.

[48] R. Landes and V. Mustonen, *Boundary value problems for strongly nonlinear second order elliptic equations*, Bolletino U.M.I. **(6) 4-B** (1985), 15-32.

[49] G. M. Liebermann, *Second Order Parabolic Differential Equations*, World Scientific, 1996.

[50] J. L. Lions, *Quelques Métodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[51] J. L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer, New York, 1972.

[52] J. D. Logan, M. R. Petersen and T. S. Shores, *Numerical study of reaction-mineralogy-porosity changes in porous media*, Appl. Math. Comput. **127** (2002), 149-164.

[53] G. Mahler, *Nonlinear parabolic problems in unbounded domains*, Proc. Roy. Soc. Edinburgh, Sect. A **82** (1978/79), 201-209.

[54] V. Mustonen, *On pseudo-monotone operators and nonlinear parabolic initial-boundary value problems on unbounded domains*, Ann. Acad. Sci. Fenn. Ser. A I. Math. **6** (1981), 225-232.

[55] V. Mustonen and C. G. Simader, *On the existence and uniqueness of solutions for strongly nonlinear elliptic variational problems*, Ann. Acad. Sci. Fenn. Ser. A I. Math. **6** (1981), 233-253.

[56] V. Mustonen, *Mappings of monotone type and nonlinear obstacle problem*, Bayreuther Math. Schriften **17** (1984), 171-183.

[57] V. Mustonen, *Mappings of monotone type: Theory and applications*, Proc. Int. Spring School "Nonlinear Analysis, Function Spaces and Applications" **4**, Teubner Texte zur Math. **119** (1990), 104-126.

[58] A. D. Myshkis, *General Theory of Differential Equations with Retarded Argument*, A.M.S. Translations, Series I, **4**, American Math. Soc., Providence, R.I., 1962.

[59] A. D. Myshkis, *Linear Differential Equations with Retarded Argument (in Russian)*, 2nd edition, Nauka, Moscow, 1972.

[60] R. Nagel, *One-parameter Semigroups of Positive Operators*, Springer, 1986.

[61] S. M. Oliva, *Boundary delay in reaction-diffusion equations: existence and convergence to equilibrium*, Journal of Dynamics and Differential Equations, **11** (1999), 279-296.

[62] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.

[63] J. Rauch, *Discontinuous semilinear differential equations and multiple valued maps*, Proc. American Math. Soc. **64** (1977), 277-282.

[64] M. Renardy, R. C. Rogers, *An Introduction to Partial Differential Equations*, Springer, 2004.

[65] M. Renardy, W. J. Hrusa and J. A. Nohel, *Mathematical Problems in Viscoelasticity*, Pitman Monographs and Surveys in Pure and Appl. Math. **35**, Longman Sci. and Techn. - John Wiley and Sons, Inc., New York, 1987.

[66] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.

[67] L. Simon, E. Baderko, *Second Order Linear Partial Differential equations* (in Hungarian), Tankönyvkiadó, Budapest, 1982.

[68] L. Simon, *Strongly nonlinear elliptic variational inequalities with nonlocal boundary conditions*, Coll. Math. Soc. J. Bolyai **53**, Qualitative Theory of Diff. Eqns., Szeged (1988), 605-620.

[69] L. Simon, *Nonlinear elliptic differential equations with nonlocal boundary conditions*, Acta Math. Hung. **56** (1990), 343-352.

[70] L. Simon, *Existence results for strongly nonlinear functional-elliptic problems*, Coll. Math. Soc. J. Bolyai **62**, Diff. Eqns., Budapest (1991), 271-287.

[71] L. Simon, *Strongly nonlinear parabolic functional differential equations*, Annales Univ. Sci. Budapest **37** (1994), 215-228.

[72] L. Simon, *On systems of strongly nonlinear parabolic functional differential equations*, Periodica Math. Hung. **33** (1996), 135-151.

[73] L. Simon, *On different types of nonlinear parabolic functional differential equations*, Pure Math. and Appl. **9** (1998), 181-192.

[74] L. Simon, *On the stabilization of solutions of nonlinear parabolic functional differential equations*, Proc. of the Conference FSDONA, Syöte (1999), 239-250.

[75] L. Simon, *Attractivity properties of solutions to a system of strongly nonlinear parabolic functional differential equations*, Math. Comp. Modelling **31** (2000), 189-197.

[76] L. Simon, *On approximation of solutions of parabolic functional differential equations in unbounded domains*, Proc. of the Conference FSDONA, Teistungen, Birkhäuser (2003), 439-451.

[77] L. Simon, *On nonlinear parabolic functional differential equations with nonlocal linear contact conditions*, Functional Diff. Eqns. **11** (2004), 153-162.

[78] L. Simon, *On contact problems for nonlinear parabolic functional differential equations*, E. J. Qualitative Theory of Diff. Eqns., Proc.7'th Coll. Qualitative Theory of Diff. Eqns., **22** (2003), 1-11.

[79] L. Simon, *On quasilinear parabolic functional differential equations of general divergence form*, Proc. of the Conference FSDONA, Milovy (2004), 280-291.

[80] L. Simon, *On quasilinear parabolic functional differential equations with discontinuous terms*, Annales Univ. Sci. Budapest **47** (2004), 211-229.

[81] L. Simon and W. Jäger, *On non-uniformly parabolic functional differential equations*, Studia Sci. Math. Hungar. **45** (2008), 285-300.

[82] L. Simon, *On qualitative properties of a system containing a singular parabolic functional equation*, EJQTDE, Proc. 8th Coll. QTDE **20** (2008), 1-13.

[83] L. Simon, *Application of monotone type operators to parabolic and functional parabolic PDE's*, Chapter 6 in Handbook on Evolutionary Differential Equations, Elsevier, 2008.

[84] L. Simon, *Nonlinear functinal parabolic equations*, Integral Methods in Science and Engineering **2**, Birkhäuser (2010), 321-326.

[85] L. Simon, *On some properties of nonlinear functional parabolic equations*, Internat J. Qualitative Theory Differential Equations Appl. **3** (2009), 140-149.

[86] L. Simon, *On some singular systems of parabolic functional equations*, Math. Bohemica **135** (2010), 123-132.

[87] A. L. Skubachevskii, *Elliptic Functional Differential Equations and Applications*, Birkhäuser, 1997.

[88] J. Tóth, P.L. Simon, *Differential Equations* (in Hungarian), Typotex, Budapest, 2005.

[89] I. I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs and Surveys in Pure and Appl. Math. **32**, Longman Sci. and Techn., Harlow (1987).

[90] J. R. L. Webb, *Boundary value problems for strongly nonlinear elliptic equations*, J. London Math. Soc. **21** (1980), 123-132.

[91] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, 1996.

[92] K. Yosida, *Functional Analysis*, Springer, 1965.

[93] E. Zeidler, *Nonlinear Functional Analysis and its Applications II A and II B*, Springer, 1990.